SCUOLA NAZIONALE DOTTORANDI DI ELETTROTECNICA
"FERDINANDO GASPARINI"
XXVII Stage

Introduction to Circuit Quantum Electrodynamics

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Università degli Studi di Napoli Federico II

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Bibliography

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- Books, Review and Research Papers cited during the Course
- Lecture Notes of the Course, https://www.nanophotonics.it/QuantumCircuits.html

Outline of Lecture 1

1.1 Introduction

1.2 A Glimpse to Superconducting circuits

1.3 Lagrangian and Hamiltonian formulations of classical mechanics

1.4 Lagrangian and Hamiltonian formulations for classical superconducting circuits

1.1 Introduction

2025 International Year of Quantum Science and Technology

The 2025 International Year of Quantum Science and Technology proclaimed by United Nation marks a century since Werner Heisenberg submitted his seminal paper 'On quantum-theoretical reinterpretation of kinematic and mechanical relationships' [1] to the Zeitschrift für Physik.

This influential work, commonly known as the 'reinterpretation' ("**Umdeutung**") paper, sought to establish a model of quantum mechanics based on relationships between quantities that are, in principle, observable.

Werner Heisenberg together with Max Born and Pascual Jordan founded Matrix Mechanics, in which physical quantities are represented through matrices, that is, operators.

[1] W. Heisenberg, Z. Physik 33, 879–893 (1925).

Circuit Quantum Electrodynamics

Circuit quantum electrodynamics [1] deals with superconducting circuits operating in the quantum regime.

[1]U. Vool, M. Devoret, Introduction to quantum electromagnetic circuits, Int. J. Circ. Theor. Appl. 2017; A. Blais et al., Circuit quantum electrodynamics, Reviews of Modern Physics 93, April-June 2021.

First hints to quantum circuits

Quantum Electrodynamic Circuits at Ultralow Temperature

Allan Widom

Department of Physics, Northeastern University, Boston, Massachusetts

(Received March 8, 1979; revised May 30, 1979)

Within present low-temperature technology it is possible to construct macroscopic circuits which exhibit quantum behavior, i.e., subcircuit currents and voltages need to be treated as operators rather than numerical quantities. The general theory of "quantum circuits" is discussed with a view toward the experimental verification of quantum electrodynamics on a macroscopic scale.

1. INTRODUCTION

It is well known that electrodynamic processes at frequency ω require quantum mechanics if the temperature is sufficiently small, $T \lesssim \hbar\omega/k_B$. With present ultralow-temperature technology, macroscopic circuits at only moderately high frequency are "quantum circuits." The nature of quantum circuits is such that voltages and currents are operators rather than numerical quantities. Circuit oscillations are "quantized" into photons.

The purpose of this work is to present the general theory of quantum circuits with a view toward the experimental verification of quantum electrodynamics on a macroscopic scale. Clearly this requires an ultralow-temperature regime.

$k_B = 1.380649 \times 10^{-23}$ J · K^{-1}

Boltzmann constant

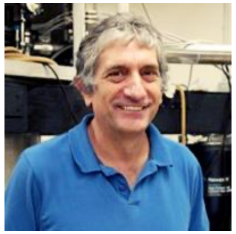
$\hbar = 1.054571817 \ldots \times 10^{-34}$ J·s^{-1}

(reduced) Planck constant

$10\ mK \iff 208.366 \ldots MHz$

A. Widom, *Quantum Electrodynamic Circuits at Ultralow Temperature*, **Journal Journal of Low Temperature Physics**, Vol. 37, Nos. 3/4, 1979.

“Do macroscopic degrees of freedom obey quantum mechanics?”



J. M. Martinis

VOLUME 55, NUMBER 15 PHYSICAL REVIEW LETTERS 7 OCTOBER 1985

Energy-Level Quantization in the Zero-Voltage State of a Current-Biased Josephson Junction

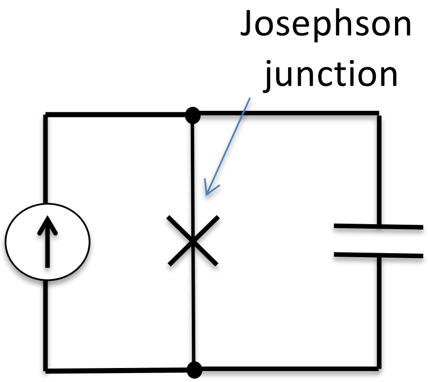
John M. Martinis, Michel H. Devoret,[(a)] and John Clarke
Department of Physics, University of California, Berkeley, California 94720, and Materials and Molecular Research Division, Lawrence Berkeley Laboratory, Berkeley, California 94720
(Received 14 June 1985)

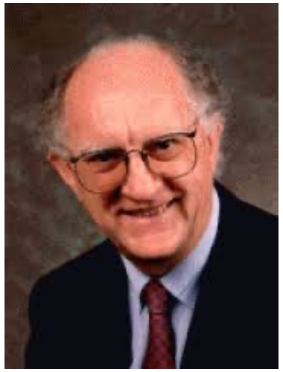
We report the first observation of quantized energy levels for a macroscopic variable, namely the phase difference across a current-biased Josephson junction in its zero-voltage state. The position of these energy levels is in quantitative agreement with a quantum mechanical calculation based on parameters of the junction that are measured in the classical regime.

PACS numbers: 03.65.−w, 05.30.−d, 74.50.+r

M. H. Devoret

Do macroscopic variables obey quantum mechanics? This question, although central to the theory of measurement,[1] has only recently been addressed experimentally. An attractive candidate for such experimental investigation is the Josephson tunnel junction, a system in which thermal fluctuations and perturbations due to the environment can be made negligible. In the case of the current-biased junction, the macroscopic variable is the phase difference, δ, between the superconducting order parameters on either side of the barrier. The junction can be represented as a particle moving in a one-dimensional tilted cosine potential.[2]





J. Clarke

[1] A. J. Legget, **Macroscopic quantum systems and the quantum theory of measurement**, Progress of Theoretical Physics Supplement, 1980.

“Macroscopic nucleus with wires”

Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction

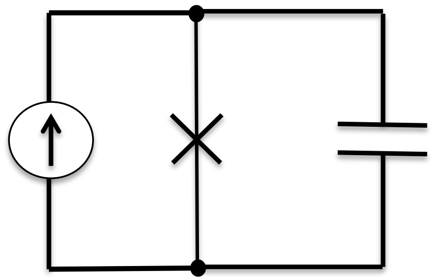
JOHN CLARKE, ANDREW N. CLELAND, MICHEL H. DEVORET, DANIEL ESTEVE, JOHN M. MARTINIS

Experiments to investigate the quantum behavior of a macroscopic degree of freedom, namely the phase difference across a Josephson tunnel junction, are described. The experiments involve measurements of the escape rate of the junction from its zero voltage state. Low temperature measurements of the escape rate for junctions that are either nearly undamped or moderately damped agree very closely with predictions for macroscopic quantum tunneling, with no adjustable parameters. Microwave spectroscopy reveals quantized energy levels in the potential well of the junction in excellent agreement with quantum-mechanical calculations. The system can be regarded as a “macroscopic nucleus with wires.”

ARE MACROSCOPIC DEGREES OF FREEDOM GOVERNED BY quantum mechanics? Our everyday experience tells us that a classical description appears to be entirely adequate. The trajectory of the center of mass of a billiard ball is predicted wonderfully well by classical mechanics. Even the Brownian motion of a tiny speck of dust in a drop of water is a purely classical phenomenon. Until recently, quantum mechanics manifested itself at the macroscopic level only through such collective phenomena as superconductivity, flux quantization, or the Josephson effect. However, these “macroscopic” effects actually arise from the coherent superposition of a large number of microscopic variables each governed by quantum mechanics. Thus, for example, the current through a Josephson tunnel junction and the phase difference across it are normally treated as classical variables. As Leggett (*1*) has emphasized, one must distinguish carefully between macroscopic quantum phenomena originating in the superposition of a large number of microscopic variables and those displayed by a single macroscopic degree of freedom. It is the latter that we discuss in this article.

Our usual observations on a billiard ball or Brownian particle reveal classical behavior because Planck's constant $\hbar$ is so tiny. However, at least in principle there is nothing to prevent us from designing an experiment in which these objects are quantum mechanical. To do so we have to satisfy two criteria: (i) the thermal energy must be small compared with the separation of the quantized energy levels, and (ii) the macroscopic degree of freedom must be sufficiently decoupled from all other degrees of freedom if the lifetime of the quantum states is to be longer than the characteristic time scale of the system (*1*). To illustrate the application of these criteria, following Leggett (*1*) we consider a simple harmonic oscillator consisting of an inductor L connected in parallel with a capacitor C. The flux Φ in the inductor and charge q on the capacitor are macroscopic conjugate variables. Observations on the oscillator are made by means of leads that unavoidably couple it to the environment. The dissipation so introduced is represented by a resistor R in parallel with L and C. The natural angular frequency of oscillation is $\omega_0 = (LC)^{-1/2}$, the impedance at the resonance frequency is $Z_0 = (L/C)^{1/2}$, and the quality factor (ratio of stored energy to energy dissipated in one oscillation) is $Q = \omega_0 CR = R/Z_0$. To observe quantum effects we thus require (i) $\hbar\omega_0 >> k_B T$, where

J. Clarke and A. N. Cleland are in the Department of Physics, University of California, and the Materials and Chemical Sciences Division, Lawrence Berkeley Laboratory, Berkeley, CA 94720. During the time these experiments were performed M. H. Devoret and J. M. Martinis were at the same address; they and D. Esteve are currently at Service de Physique, Centre d'Études Nucléaires de Saclay, 91191 Gif-sur-Yvette Cedex, France.



Science, vol. 239, **1988**

Circuit Quantum Electrodynamics

Circuit quantum electrodynamics [1] deals with superconducting circuits operating in the quantum regime.

Superconducting quantum circuits represent one of the most promising pathways for developing digital quantum computing [2], analogic simulation of quantum systems [3] and optimization using quantum annealing [4].

[1]U. Vool, M. Devoret, Introduction to quantum electromagnetic circuits, Int. J. Circ. Theor. Appl. 2017; A. Blais et al., Circuit quantum electrodynamics, Reviews of Modern Physics 93, April-June 2021.

[2] M. Nielsen, I. Chuang, Quantum Computation and Quantum Information (10th anniversary ed.) Cambridge University press, 2010.

[3] I. Georgescu, S. Ashhab, F. Nori, Quantum Simulation, Reviews of Modern Physics, 2014.

[4] B. K. Chakrabarti et al., Quantum Annealing and Computation: Challenges and Perspectives". Philosophical Transactions A. 381, 2023.

Digital Computation

- Classical digital computers are deterministic, they determine a specific singular outcome for any input.
- Quantum digital computers are probabilistic, finding the most likely solution to a problem.

For extremely complex problems that quantum computers could eventually tackle (chemistry, cybersecurity, data analytics and artificial intelligence, optimization and simulation, data management and searching), probabilistic approach might dramatically reduce computation time by hundreds of thousands of years compared to conventional techniques implemented on classical computers.

Classical Bit versus Qubit

- Classical bits exist in one of two distinct states, e.g., two distinct voltage values of digital devices.

- Qubits or quantum bits are the quantum version of classical bits (basic unit of quantum information). They are realized through two-level quantum systems.

The classical bit is in one state or the other, instead, quantum mechanics allows the qubit to be in a coherent superposition of two states simultaneously.

Quantum Digital Computer

A set of qubits forms a qubit register.

Quantum computers perform calculations by manipulating the quantum states of qubits within the register.

Many possible outcomes are canceled out through interference, while others are amplified. The amplified outcomes are the most likely solutions to the problem that is object of the calculations.

Qubit

Any two-level quantum mechanical system can implement a qubit:

- artificial atoms made with superconducting quantum circuits, with energies E_0 and E_1,

$|1\rangle$(ket 1) ——— E_1

$$E_0 < E_1$$

$|0\rangle$(ket 0) ——— E_0

- intrinsic magnetic moment of electrons in which the two levels can be taken as the magnetic moment up and down;

- polarization of single photons in which the two level can be taken as the horizontal and vertical linear polarizations of light.

Superposition of States

- artificial atoms made with superconducting quantum circuits, with energies E_0 and E_1,

$|1\rangle$(ket 1) ———————— E_1

$E_0 < E_1$

$|0\rangle$(ket 0) ———————— E_0

$|0\rangle$ is the state of the atom for which the measurement of energy yields certainly the value E_0 and $|1\rangle$ is the state for which the measurement of energy yields certainly the value E_1.

Superposition of States

- artificial atoms made with superconducting quantum circuits, with energies E_0 and E_1,

$|1\rangle$(ket 1) ——— E_1

$E_0 < E_1$

$|0\rangle$(ket 0) ——— E_0

$|0\rangle$ is the state of the atom for which the measurement of energy yields certainly the value E_0 and $|1\rangle$ is the state for which the measurement of energy yields certainly the value E_1.

The state of the atom can also be a linear superposition of the states $|0\rangle$ and $|1\rangle$,

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle,$$

where c_0 and c_1 are complex numbers with $|c_0|^2 + |c_1|^2 = 1$.

Superposition of States

- artificial atoms made with a superconducting quantum circuit, with two values of energy E_0 and E_1,

$|1\rangle$(ket 1) ———————— E_1

$E_0 < E_1$

$|0\rangle$(ket 0) ———————— E_0

In general, the state of the atom is a linear superposition of the states $|0\rangle$ and $|1\rangle$,

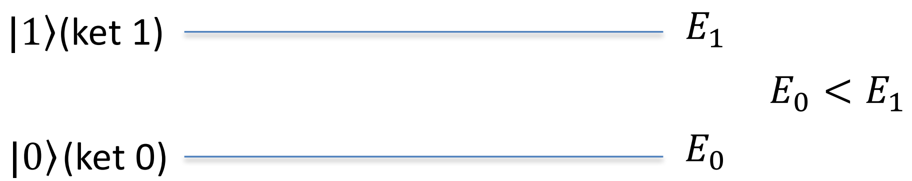
$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

where c_0 and c_1 are complex numbers with $|c_0|^2 + |c_1|^2 = 1$.

- $|\boldsymbol{c_0}|^2$ yields the probability that the outcome of the measurement of energy is E_0;
- $|\boldsymbol{c_1}|^2$ yields the probability that the outcome of the measurement of energy is E_1.

Information is represented by the statistics.

Quantum Measurement and Collapse of State



$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

If the result of the measurement of energy of the atom is E_0 the quantum state $|\psi\rangle$ of the atom collapses after the measurement to the state $e^{i\varphi_0}\,|0\rangle$, and if the result of the measurement of energy is E_1 the quantum state of the atom $|\psi\rangle$ collapses after the measurement to the state $e^{i\varphi_1}\,|1\rangle$, where the phase factors are not experimentally accessible.

The quantum measurement is an irreversible process.

Qubit Physical Implementation

More common types of qubits in use are:

Artificial atoms based on superconducting quantum circuit made with superconducting metals operating at temperatures of the order of $10\ mK$. These qubits are favored for their speed in performing computations and fine-tuned control.

Trapped ions are noted for the long coherence times and high-fidelity measurements.

Quantum dots are small semiconductors that capture a single electron and use it as a qubit. They offer promising potential for scalability and compatibility with existing semiconductor technology.

Photons are individual light particles used to send quantum information across long distances through optical fiber cables; they are currently being used in quantum communication and quantum cryptography.

In **Neutral atoms** qubits are encoded in the energy levels and are controlled with lasers. They are well suited for scaling and performing logical operations.

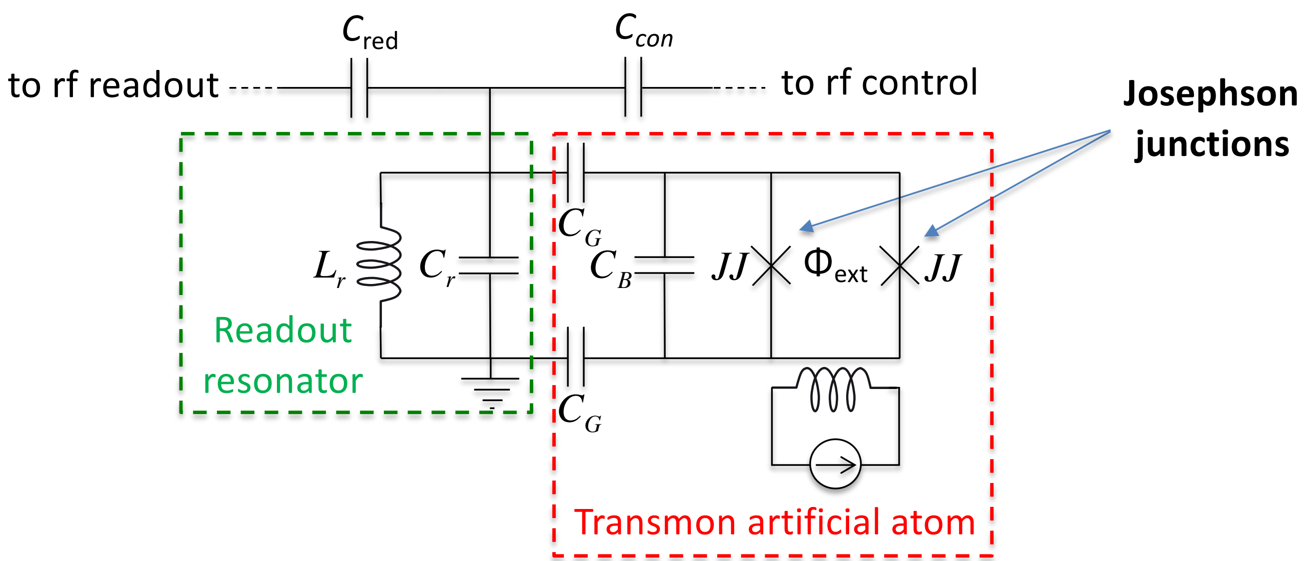
Ze-Liang Xiang, Sahel Ashhab, J. Q. You, Franco Nori, Hybrid quantum circuits: Superconducting circuits interacting with other quantum systems, Rev. Mod. Phys. 85, 623 – Published 9 April, 2013.

Superconducting Quantum Circuits

Trapped ions, neutral atoms, electron spins in silicon and quantum dots, polarized photons, … encode quantum information in microscopic elements, such as ions, atoms, electrons or photons.

Superconducting quantum circuits are quite different: they are macroscopic in size, and they are printed lithographically on wafers like classical computer chips. Their quantum features such as energy spectra, superpositions of states, transition probabilities, interference, entanglement, coupling strengths, coherence rates depend on macroscopic circuit parameters. Consequently, it is possible to design superconducting quantum circuits so that they display specific quantum mechanical behaviors.

Superconducting Quantum Circuit for Quantum Computing



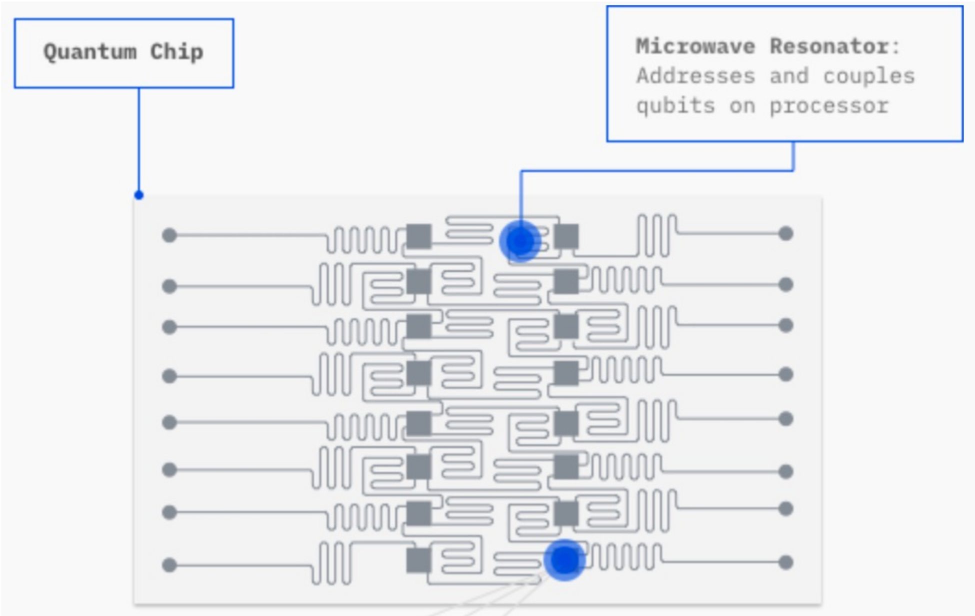
The transmon, which can implement a qubit, is realized through a nonlinear LC circuit.

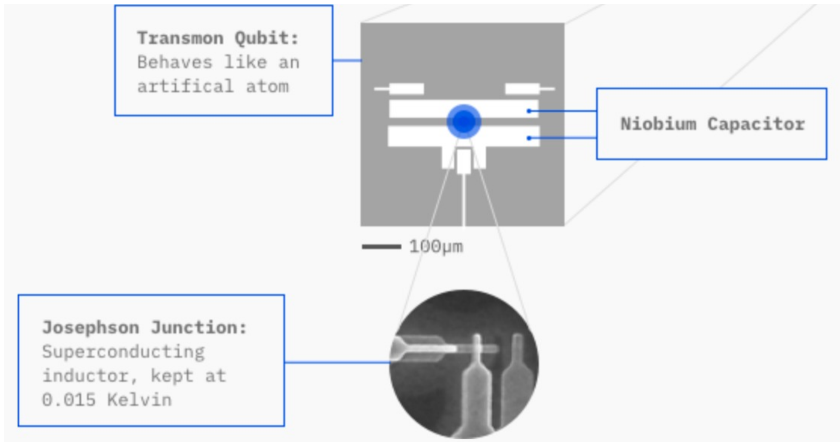
The readout resonator, used to measure the transmon quantum state, consists of a linear LC circuit.

By firing coherent microwave signal, it is possible to control the qubit behavior and read its quantum state.

By courtesy of Alessandro Miano

Quantum Digital Processor





Credit: IBM Research

Superconducting Circuit - Based Quantum Processors

Quantum Digital Processor

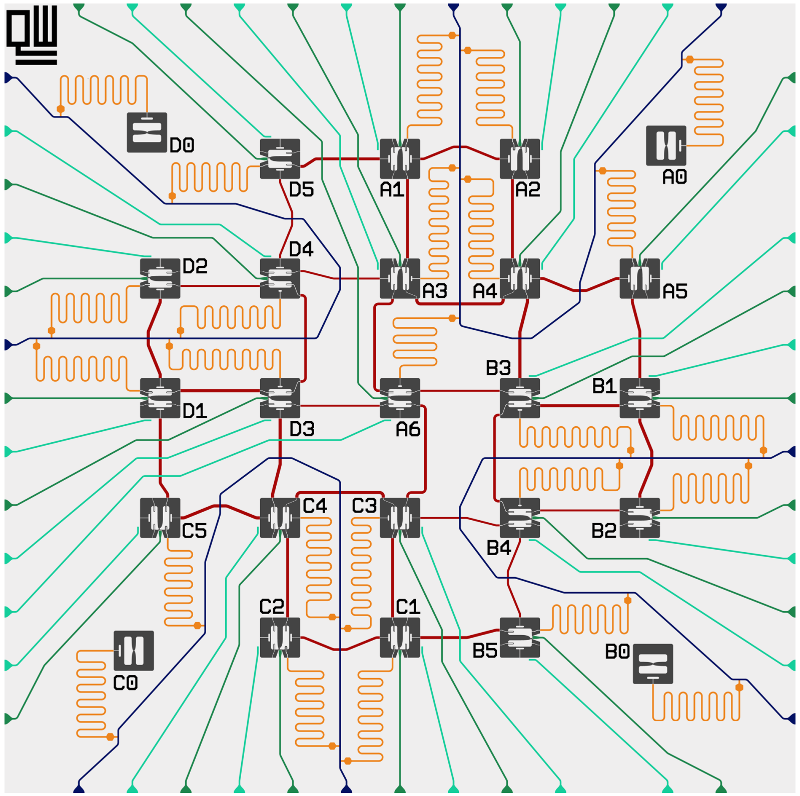
- IBM 1.121 Qubit (physical) December 2023 (USA)
- Chinese Academic of Science 504 Qubit (physical) 2024 (China)
- Google 105 Qubit (physical) December 2024 (USA)
- Rigetti 84 Qubit (physical) December 2023 (USA)
- University of Science and Technology of China 105 Qubit (physical) December 2024 (China)
- Quantware 64 Qubit (physical) February 2023 (Delft, Europe)
- …

Annealing Quantum Processors

- D-Wave 7.440 Qubit 2024 (Canada)

25 - Qubit Quantum Processor in Naples

(Dipartimento di Fisica “Ettore Pancini”)



Credit: Quantware

1.2 A glimpse to superconducting circuits

Superconducting circuits

The toolbox of elements of superconducting circuits consists of linear elements such as capacitors, inductors, coupled circuits, transmission lines, distributed resonators and nonlinear elements based on Josephson junctions [1].

They are made with low temperature superconducting metals [1] (Type I superconductors): aluminum (critical temperature $1.18\ K$), niobium (critical temperature $9.25\ K$), … .

[1] T. Orlando, K. A. Delin, Foundations of Applied Superconductivity, Addison Wesley (1991).

Cryostat: Cabled Dilution Refrigerator

Actually, superconducting circuits necessitate millikelvin temperatures.

These low temperatures are essential for initializing circuits in their ground state and minimizing errors caused by thermal excitation during operation.

Such temperatures are attained using dilution refrigerators.

The pressure in the cryostat is below 10^{-5} mbar.

a) b)

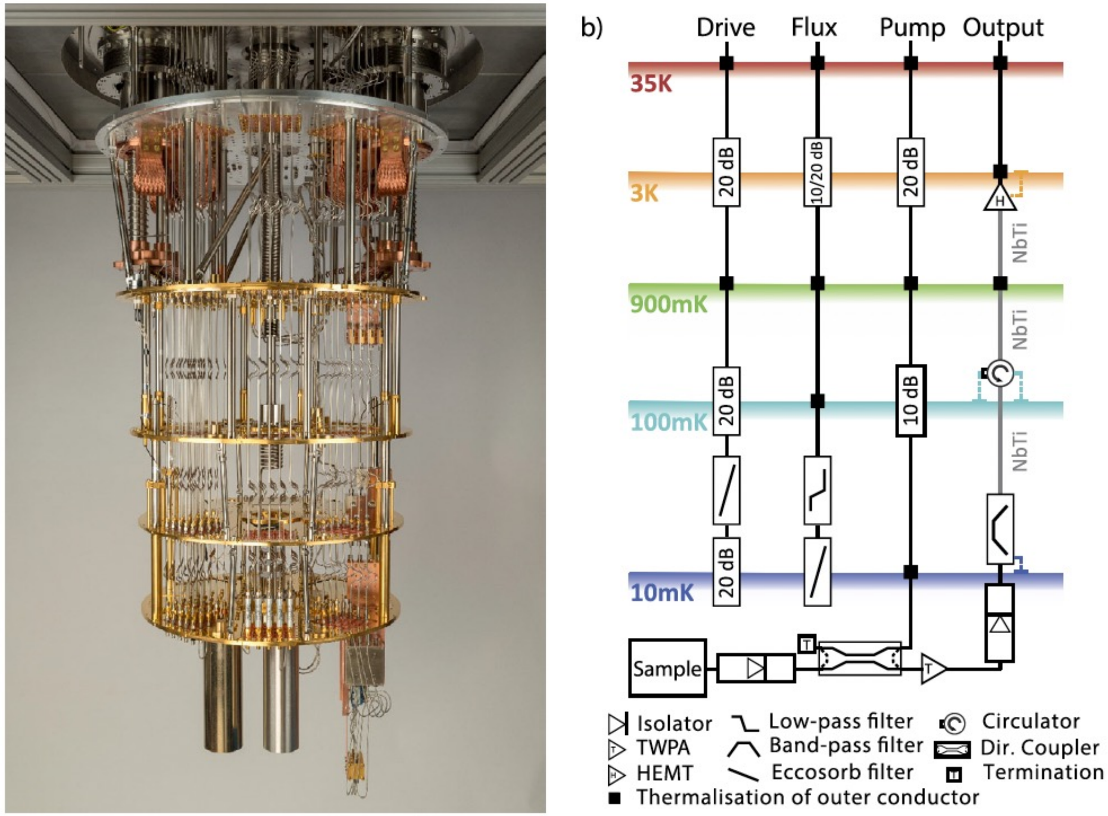
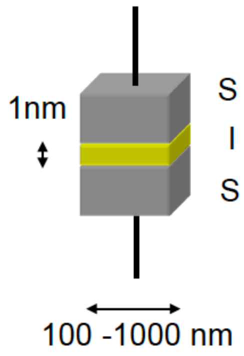


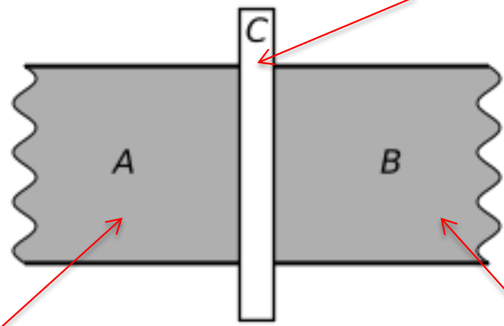
Figure 3 Cabled dilution refrigerator (DR). (**a**) Bluefors XLD DR with 25 drive lines, 25 flux lines, 4 read-out, 6 read-in, and 5 pump lines installed (see end of Sect. 3.1 for details). (**b**) Schematic of the cabling configuration within the DR

S. Krinner *et al.*, Engineering cryogenic setups for 100 qubit scale superconducting circuit systems, EPJ Quantum Technology, 2019.

Fundamental Nonlinear Element: Josephson Junction



gap of an insulating material

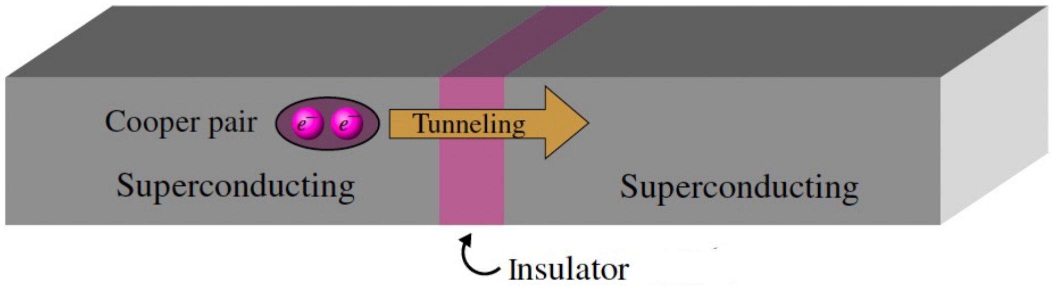


Superconducting electrode
(for instance, Aluminum)

Superconducting electrode
(for instance, Aluminum)

In low temperature superconducting metals free electrons are paired in Cooper pairs.

In Josephson junctions Cooper pairs cross the insulating thin barrier through the tunneling effect.

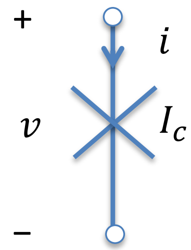


A. Barone, G. Paternò, *Physics and Applications of the Josephson Effect*, John Wiley & Sons, Inc.

30

Josephson Junction macroscopic behaviour: ideal one-port

$$\begin{cases} v = \dfrac{\Phi_0}{2\pi}\dfrac{d\theta}{dt} \\ i = I_c \sin(\theta) \end{cases}$$



θ is the **Josephson phase** between the two superconducting electrodes of the junction.

- ✓ $\Phi_0 = 2.0678 \ldots \times 10^{-15}\ weber$ is the «**quantum of superconducting magnetic flux**».
- ✓ I_c is the **critical current** of the junction: a macroscopic parameter proportional to the area of the junction and "transparency" of the tunnel barrier; typical values of I_c are in the $\mu A \div nA$ range .

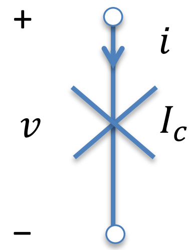
A. Barone, G. Paternò, *Physics and Applications of the Josephson Effect*, John Wiley & Sons, Inc.

Josephson Junction ideal one-port

characteristic relations

$$\begin{cases} v = \dfrac{d\phi}{dt} \\ i = I_c \sin(2\pi \dfrac{\phi}{\Phi_0}) \end{cases}$$

$\phi \equiv \frac{\Phi_0}{2\pi}\theta$ "effective (kinetic) flux"

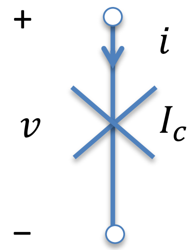


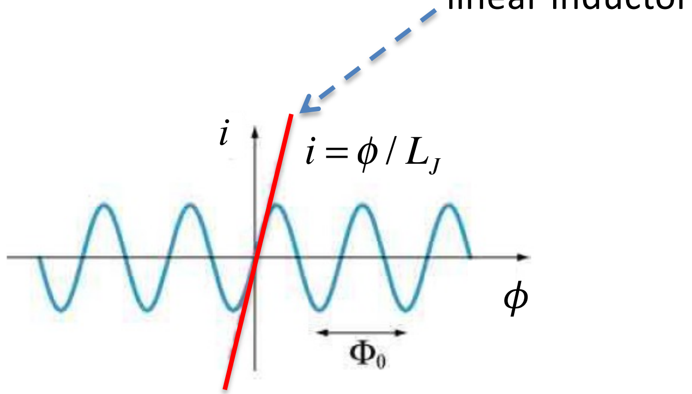
The Josephson junction behaves as a nonlinear inductor

Josephson Junction ideal one-port

$$\begin{cases} v = \dfrac{d\phi}{dt} \\ i = I_c \sin(2\pi \dfrac{\phi}{\Phi_0}) \end{cases}$$

$\phi \equiv \frac{\Phi_0}{2\pi}\theta$ "effective (kinetic) flux"





$$L_J = \frac{1}{2\pi}\frac{\Phi_0}{I_c} \approx 1\mu H \div 1nH$$

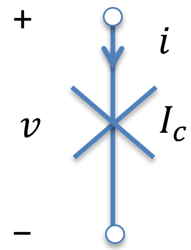
"kinetic" inductance of the Josephson junction

The linearized one-port around $\phi = 0$ is equivalent to a linear inductor with inductance L_J

Josephson Junction ideal one-port

$$\begin{cases} v = \dfrac{d\phi}{dt} \\ i = I_c \sin(2\pi \dfrac{\phi}{\Phi_0}) \end{cases}$$

$\phi \equiv \frac{\Phi_0}{2\pi}\theta$ "effective (kinetic) flux"

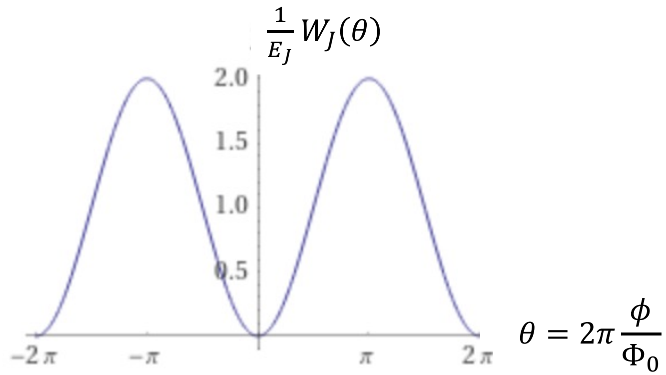


Stored energy in the Josephson junction

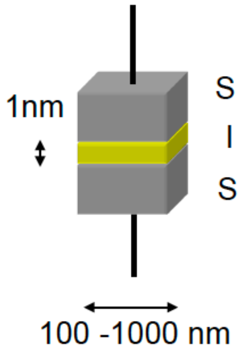
$$W_J(\phi) = \int_0^{\phi} i(\phi')d\phi' = E_J[1 - cos(2\pi\phi/\Phi_0)]$$

$$E_J = \frac{1}{2\pi} I_c \Phi_0 \cong 10\mu eV \div 10meV$$

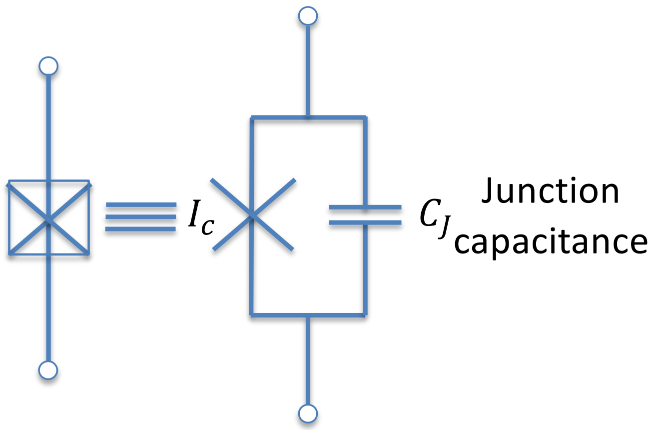
$$1\ eV = 1.60 \ldots . \times 10^{-19}\ \mathrm{J}$$



Equivalent scheme of a lossless Josephson Junction



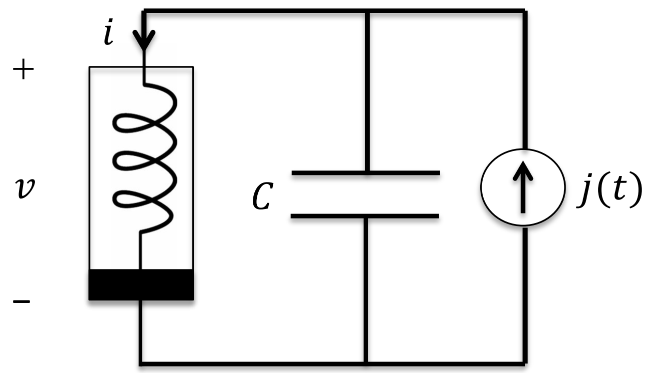
Superconductor-insulator-superconductor



The values of the junction capacitance depend on the junction area and insulating thickness. In experiments, typical values are in the $pF - fF$ range.

A Simple Superconducting Circuit: Classical Description

$i = i(\phi)$



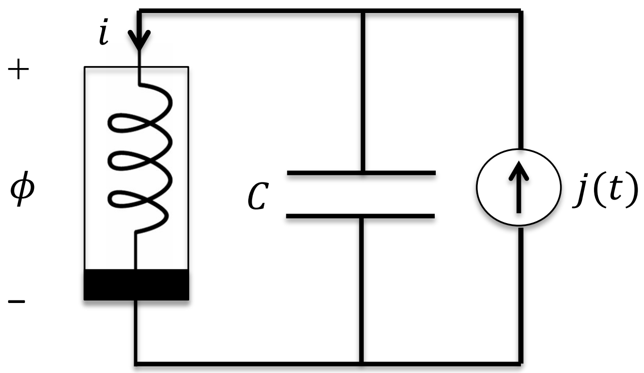
It is convenient to choose the flux ϕ as degree of freedom of the circuit,

$$\phi(t) = \int_{-\infty}^{t} v(\tau)d\tau\,.$$

$i = i(\phi)$ is the characteristic of the nonlinear inductor

A Simple Superconducting Circuit: Classical Description

$i = i(\phi)$

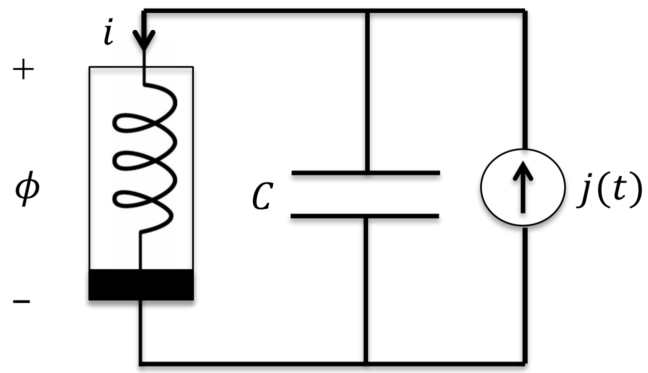


The equation governing the flux is

$$C\frac{d^2\phi}{dt^2} + i(\phi) = j(t)$$

A Simple Superconducting Circuit: Classical Description

$i = i(\phi)$



$$W_I(\phi) = \int_0^{\phi} i(\phi')d\phi'$$

stored energy in the nonlinear inductor

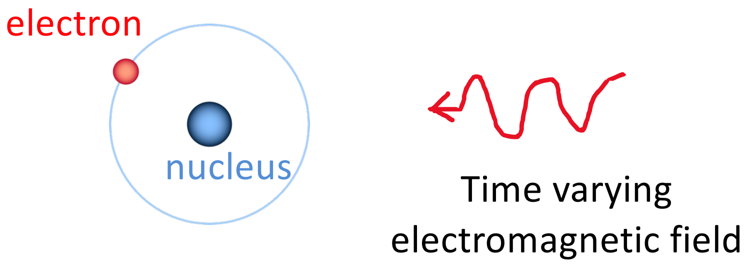
$$i(\phi) = \frac{dW_I}{d\phi}$$

The equation governing the flux is

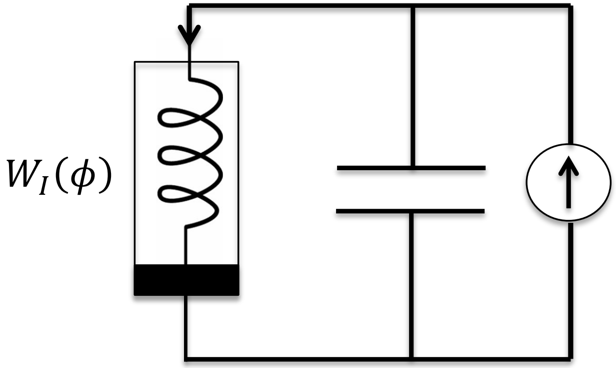
$$C\frac{d^2\phi}{dt^2} + \frac{dW_I}{d\phi} = j(t)$$

Artificial atom

Atom

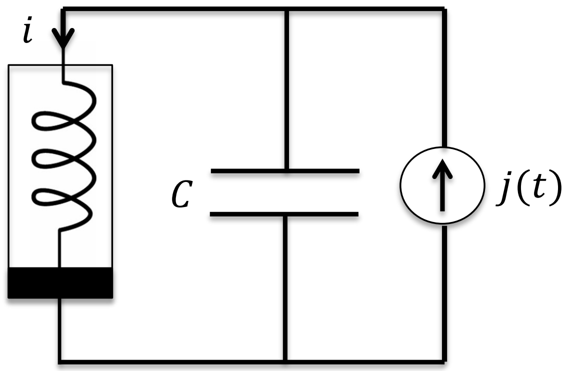


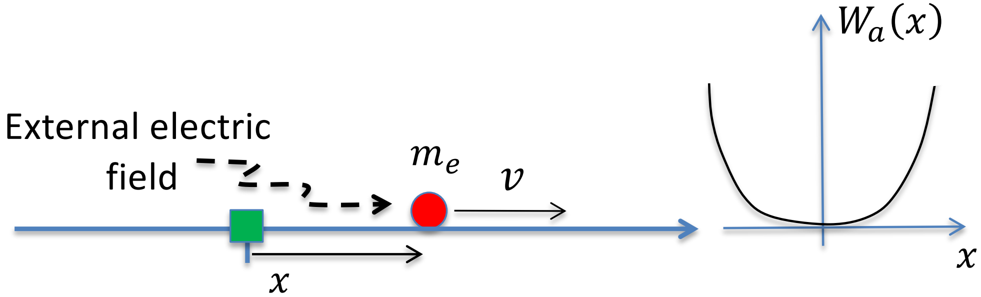
Artificial atom: «macroscopic nucleus with wires»



The nonlinear LC superconducting circuit, in principle, can emulate the behaviour of the electron in a hydrogen atom.

Superconducting circuit versus electron in 1-D potential well: classical description





Electron in a potential well $W_a(x)$
under the action of an external electric field

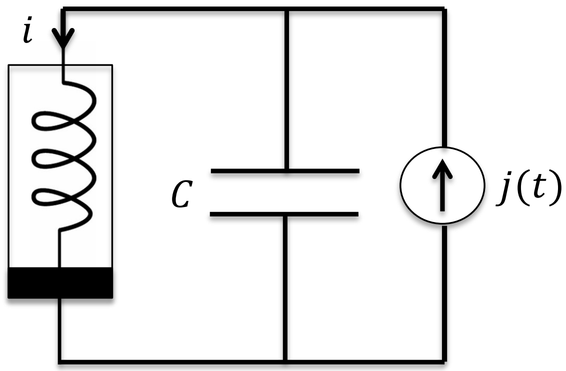
$$C\frac{d^2\phi}{dt^2} + \frac{dW_I}{d\phi} = j(t)$$

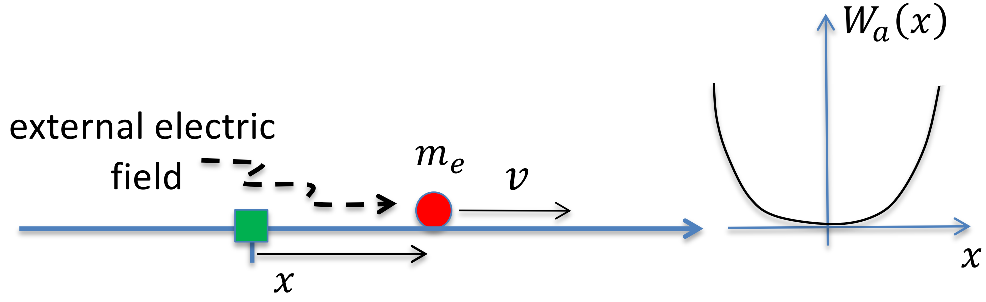
$$m_e\frac{d^2x}{dt^2} = -\frac{dW_a}{dx} + F(t)$$

Force due to the well

Force due to the external electric field

Superconducting circuit versus electron in 1-D potential well: classical description





Electron in a potential well $W_a(x)$
under the action of an external electric field

$$C\frac{d^2\phi}{dt^2} + \frac{dW_I}{d\phi} = j(t)$$

Macroscopic system

$$m_e\frac{d^2x}{dt^2} + \frac{dW_a(x)}{dx} = F(t)$$

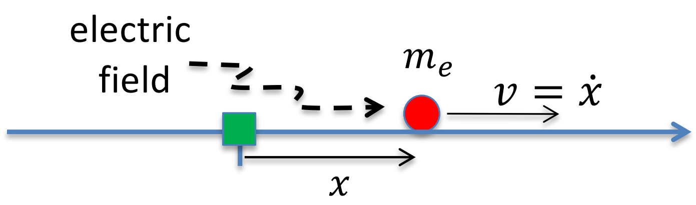
Microscopic system

The behaviour of the superconducting circuit is analogous to that of the electron in 1-D potential well, both in the classical and quantum regime.

1.3 Lagrangian and Hamiltonian formulations of classical mechanics

L.D. Landau & E.M. Lifshitz, *Mechanics* (***A Course of Theoretical Physics***, Volume 1) Pergamon Press.

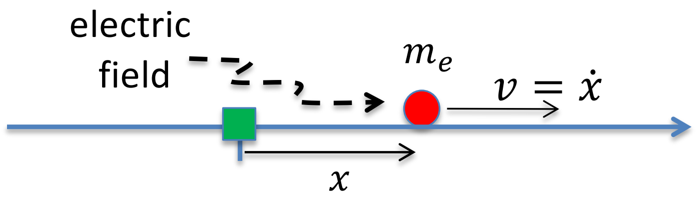
An electron with one degree of freedom



Kinetic energy of the electron (non-relativistic) $K(\dot{x}) = \frac{1}{2} m_e \dot{x}^2$

Potential energy of the electron $W_a = W_a(x)$

Lagrangian Formulation



Kinetic energy of the electron (non-relativistic) $K(\dot{x}) = \frac{1}{2} m_e \dot{x}^2$

Potential energy of the electron $W_a = W_a(x)$

System Lagrangian $L(x, \dot{x}; t) = K(\dot{x}) - W_a(x) + F(t)x$

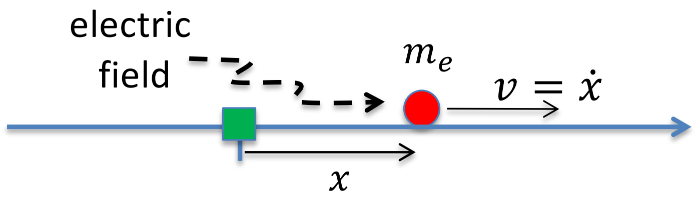
Kinetic energy

Potential energy

Contribute
due to the external field

L.D. Landau & E.M. Lifshitz, *Mechanics* (***A Course of Theoretical Physics***, Volume 1) Pergamon Press.

Lagrangian Formulation

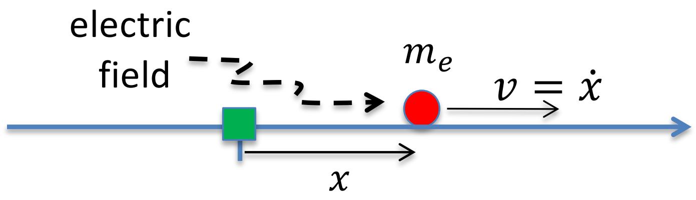


Circuit Lagrangian: $L(x, \dot{x}; t) = K(\dot{x}) - W_a(x) + F(t)x$

Action of the Electron on the time interval (t_1, t_2): $S\{x(t)\} = \int_{t_1}^{t_2} L[x(t), \dot{x}(t); t]dt$

The action is a functional of x(t)

Lagrangian Formulation



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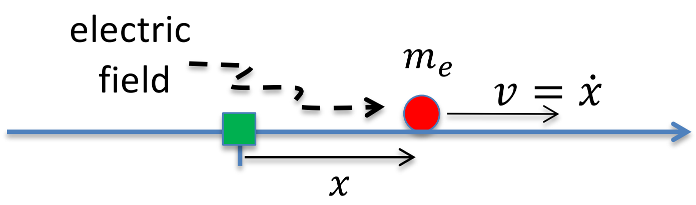
The action is a functional of x(t)

The Planck constant h, which is of foundational importance in quantum physics, is the quantum of action,

$h = 6.62607015 \times 10^{-34}$ J$\cdot s$ (exact value)

Reduced Planck constant $\hbar = \frac{h}{2\pi} = 1.054571817... \times 10^{-34}$ J$\cdot s$

Lagrangian Formulation



Circuit Lagrangian: $L(x, \dot{x}; t) = K(\dot{x}) - W_a(x) + F(t)x$

Action of the electron on the time interval (t_1, t_2): $S\{x(t)\} = \int_{t_1}^{t_2} L[x(t), \dot{x}(t); t]dt$

The action is a functional of x(t)

$\hbar = 1.054571817... \times 10^{-34}$ J$\cdot s$

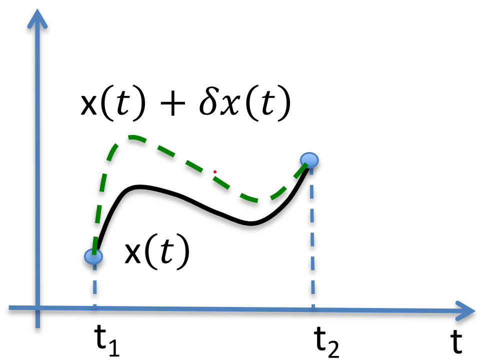
When the action of the electron is of the order of the Planck's constant,
quantum behaviour are important.

Principle of Least Action

The particle moves in any interval (t_1, t_2) in such a way that the action is <u>stationary</u> , *i.e.*,

$$\delta S = S\{x(t) + \delta\phi(t)\} - S\{x(t)\} = 0$$

for any $\delta x(t)$ with $\delta x(t_1) = \delta x(t_2) = 0$.



$$S\{x(t)\} = \int_{t_1}^{t_2} L[x(t), \dot{x}(t); t]dt$$

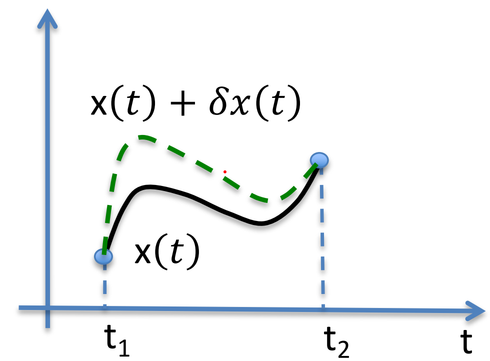
L.D. Landau & E.M. Lifshitz, *Mechanics* (***A Course of Theoretical Physics***, Volume 1) Pergamon Press.

Euler – Lagrange equation

The particle moves in any interval (t_1, t_2) in such a way that the action is <u>stationary</u> , *i.e.*,

$$\delta S = S\{x(t) + \delta\phi(t)\} - S\{x(t)\} = 0$$

for any $\delta x(t)$ with $\delta x(t_1) = \delta x(t_2) = 0$.



$$S\{x(t)\} = \int_{t_1}^{t_2} L[x(t), \dot{x}(t); t]dt$$

The action is stationary if its **first variation** with respect to $x(t)$ is equal to zero.

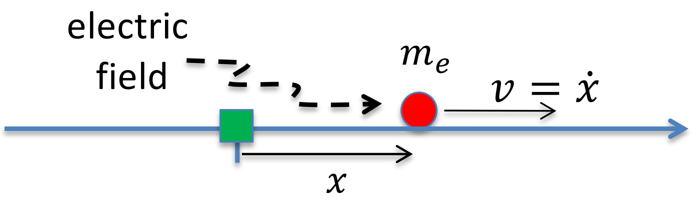
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$ **Euler – Lagrange equation of the particle**

Euler – Lagrange equation

$$L(x, \dot{x}; t) = \frac{1}{2} m_e \dot{x}^2 - W_a(x) + F(t)x$$

It is easy to verify that $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$ ⟹ $m_e \frac{d^2 x}{dt^2} + \frac{dW_a(x)}{dx} = F(t)$

Conjugate Mechanical Variables



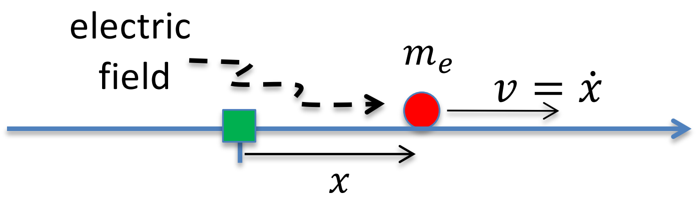
Lagrangian $L(x, \dot{x}; t) = \frac{1}{2} m_e \dot{x}^2 - W_a(x) + F(t)x$

Euler-Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$

Conjugate momentum $p = \frac{\partial L}{\partial \dot{x}} = m_e \dot{x}$

(x, p) are said to be **canonical conjugate variables**: they are state variables of the system

Conjugate Mechanical Variables



Lagrangian $L(x, \dot{x}; t) = \frac{1}{2} m_e \dot{x}^2 - W_a(x) + F(t)x$

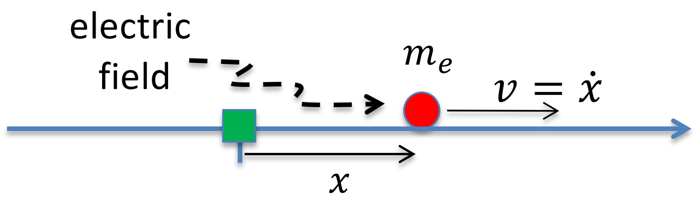
Euler-Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$

Conjugate momentum $p = \frac{\partial L}{\partial \dot{x}} = m_e \dot{x}$

(x, p) are said to be **canonical conjugate variables**: they are state variables of the system

Canonical conjugate physical variables play a very important role:
in the quantum regime they cannot be measured at the same time with the same precision,
Heisenberg Uncertainty Principle

Hamiltonian Formulation



We require a model that treats position and conjugate momentum on equal footing.

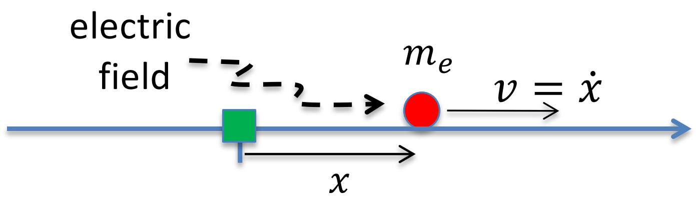
Lagrangian $L(x, \dot{x}; t) = \frac{1}{2} m_e \dot{x}^2 - W_a(x) + F(t)x$

Conjugate momentum $p = \frac{\partial L}{\partial \dot{x}}$

Hamiltonian of the system $H = (p\dot{x} - L)$

L.D. Landau & E.M. Lifshitz, *Mechanics* (***A Course of Theoretical Physics***, Volume 1) Pergamon Press.

Hamiltonian Formulation



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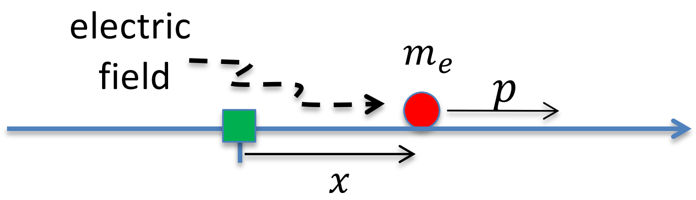
Lagrangian $L(x, \dot{x}; t) = \frac{1}{2} m_e \dot{x}^2 - W_a(x) + F(t)x$

Conjugate momentum $p = \frac{\partial L}{\partial \dot{x}} = m_e \dot{x} \Longrightarrow \dot{x} = \frac{1}{m_e} p$

Hamiltonian of the system $H = p\dot{x} - L$

Hamiltonian Formulation



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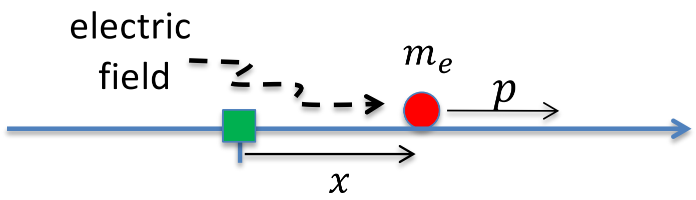
Lagrangian $L(x, \dot{x}; t) = \frac{1}{2} m_e \dot{x}^2 - W_a(x) + F(t)x$

Conjugate momentum $p = \frac{\partial L}{\partial \dot{x}} = m_e \dot{x} \Longrightarrow \dot{x} = \frac{1}{m_e} p$

Hamiltonian of the system $H(p, x; t) = p\dot{x} - L = \frac{1}{2m_e} p^2 + W_a(x) - F(t)x$

Hamiltonian Formulation



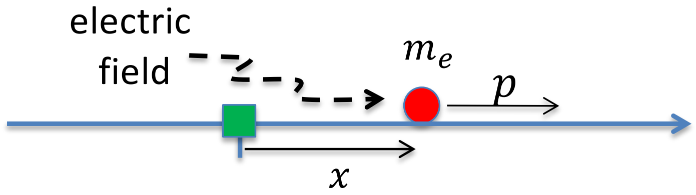
Hamiltonian of the system $H(p, x; t) = [\frac{1}{2m_e}p^2 + W_a(x)] - F(t)x$

Hamilton equations
$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}, \\ \dot{p} = -\dfrac{\partial H}{\partial x}. \end{cases}$$

Verify that the conjugate variables are governed by the Hamilton equations.

L.D. Landau & E.M. Lifshitz, *Mechanics* (***A Course of Theoretical Physics***, Volume 1) Pergamon Press.

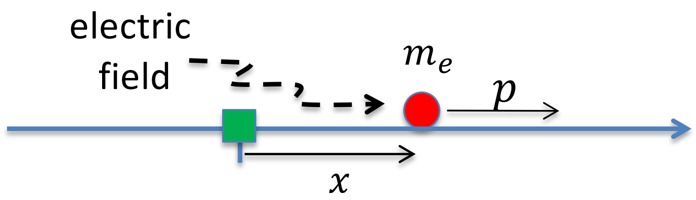
Hamiltonian Formulation



Hamiltonian of the system $H(p,x;t) = [\frac{1}{2m_e}p^2 + W_a(x)] - F(t)x$

Hamilton equations
$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}, \\ \dot{p} = -\dfrac{\partial H}{\partial x}. \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x} = \dfrac{1}{m_e}p, \\ \dot{p} = -\dfrac{dW_a}{dx} + F(t) \end{cases}$$
State equation of the electron

Hamiltonian Formulation



Hamiltonian of the system $H(p,x;t) = [\frac{1}{2m_e}p^2 + W_a(x)] - F(t)x$

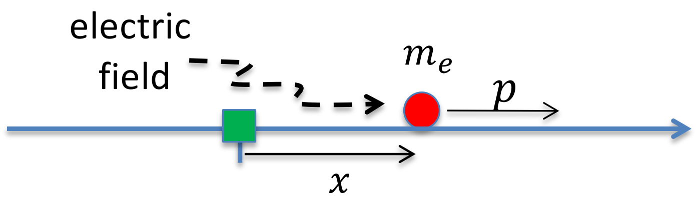
particle energy

Applied field contribution

Fundamental property of the Hamiltonian

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -x\frac{dF}{dt}$$

Hamiltonian Formulation



Hamiltonian of the system $H(p, x; t) = [\frac{1}{2m_e} p^2 + W_a(x)] - F(t)x$

particle energy

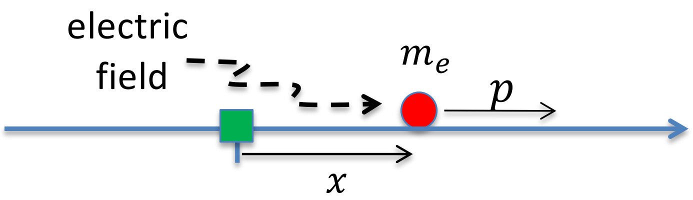
Applied field contribution

Fundamental property of the Hamiltonian

$$\frac{dH}{dt} = 0 \text{ if } F(t) = 0$$

When $F(t) = 0$ the Hamiltonian is a constant of motion: in this case it coincides with the energy of the particle, which is conserved because the system is isolated.

Hamiltonian Formulation



Hamiltonian of the system $H(p,x;t) = [\frac{1}{2m_e}p^2 + W_a(x)] - F(t)x$

The Hamiltonian plays a fundamental role in Quantum Mechanics.

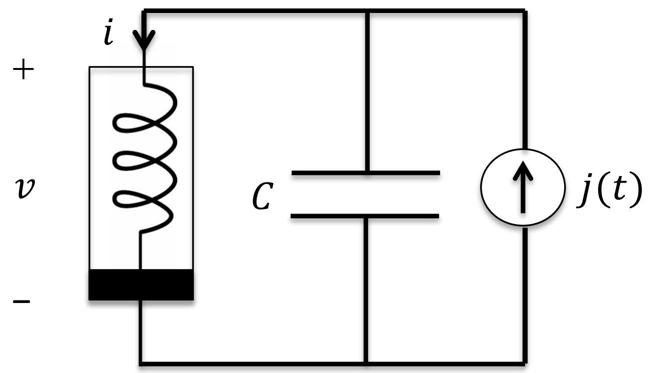
The Hamiltonian is the physical variable on which the Schrödinger equation is based: the Schrödinger governs the time evolution of the quantum state of the particle.

1.4 Lagrangian and Hamiltonian formulations for classical superconducting circuits

A Simple Superconducting Circuit

Energy stored
in the nonlinear inductor

$W_I = W_I(\phi)$



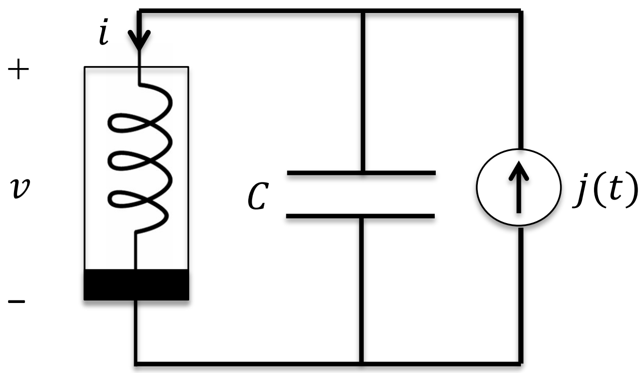
Energy stored
in the capacitor

$W_c = \frac{1}{2} C v^2 = \frac{1}{2} C \dot{\phi}^2$

Lagrangian Formulation

Energy stored
in the nonlinear inductor

$W_I = W_I(\phi)$



Energy stored
in the capacitor

$W_c = \frac{1}{2} C v^2 = \frac{1}{2} C \dot{\phi}^2$

Circuit Lagrangian: $L(\phi, \dot{\phi}; t) = \frac{1}{2} C \dot{\phi}^2 - W_I(\phi) + j(t)\phi$

Wells, D.A.. *Application of the Lagrangian equations to electrical circuits*. **Journal of Applied Physics**, 9(5), 312–320 (1938).

Chua, L. and McPherson, J. *Explicit topological formulation of Lagrangian and Hamiltonian equations for nonlinear networks*. **IEEE Transactions on Circuits and Systems** , 21(2), 277–286 (1974).

Lagrangian Formulation

Energy stored
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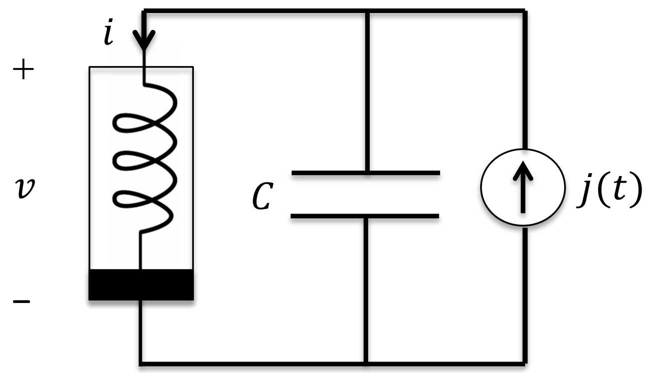
$$W_I = W_I(\phi)$$



Energy stored
in the capacitor

$$W_c = \frac{1}{2} C v^2 = \frac{1}{2} C \dot{\phi}^2$$

Circuit Lagrangian: $L(\phi, \dot{\phi}; t) = \frac{1}{2} C \dot{\phi}^2 - W_I(\phi) + j(t)\phi$

Circuit action on the time interval (t_1, t_2): $S\{\phi(t)\} = \int_{t_1}^{t_2} L[\phi(t), \dot{\phi}(t); t] dt$

The action is a functional of $\phi\ (t)$

Lagrangian Formulation

Energy stored
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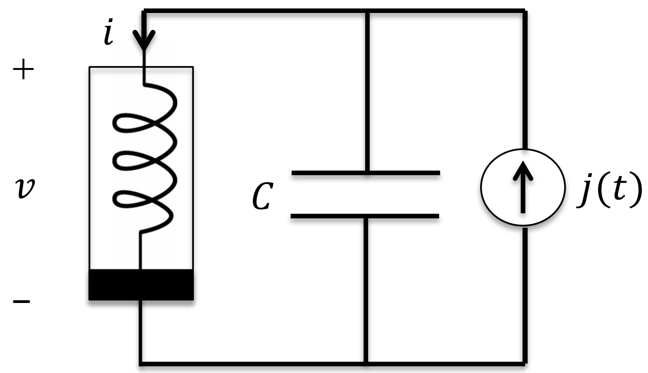

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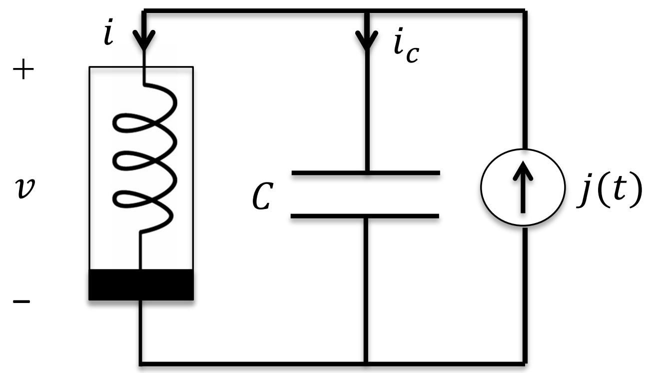
The action is a functional of $\phi\ (t)$

Principle of least action → $\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$ Euler-Lagrange equation

Lagrangian Formulation

Energy stored
in the nonlinear inductor

$$W_I = W_I(\phi)$$



Energy stored
in the capacitor

$$W_c = \frac{1}{2}Cv^2 = \frac{1}{2}C\dot{\phi}^2$$

$$L(\phi, \dot{\phi}; t) = \frac{1}{2}C\dot{\phi}^2 - W_I(\phi) + j(t)\phi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$ Euler-Lagrange equation

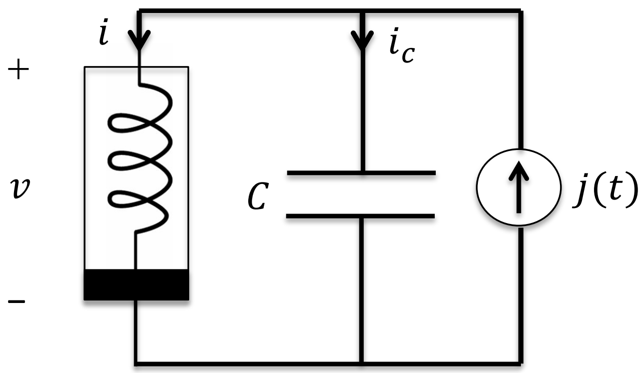
$$i_c + i - j = 0$$ Current Kirchhoff law

$$C\frac{d^2\phi}{dt^2} + \frac{dW_I}{d\phi} - j(t) = 0$$

Canonically conjugate variables of the circuit

Energy stored in the nonlinear inductor

$$W_I = W_I(\phi)$$



Energy stored in the capacitor

$$W_c = \frac{1}{2}Cv^2 = \frac{1}{2}C\dot{\phi}^2$$

$$L(\phi, \dot{\phi}; t) = \frac{1}{2}C\dot{\phi}^2 - W_I(\phi) + j(t)\phi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$ Euler-Lagrange equation

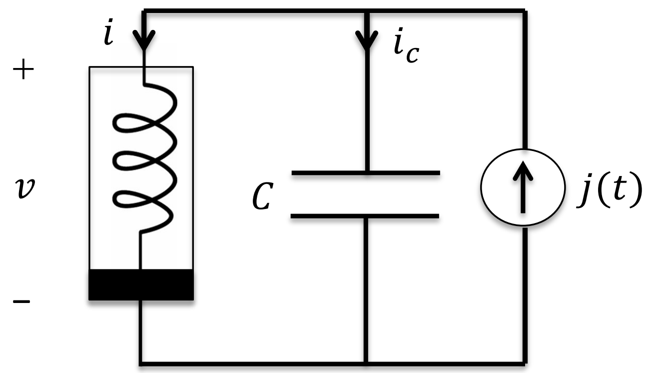
Conjugate momentum $q = \frac{\partial L}{\partial \dot{\phi}}$

(ϕ, q) are **canonical conjugate variables** of the circuit: they are state variables of the system

Canonically conjugate variables of the circuit

Energy stored
in the nonlinear inductor

$$W_I = W_I(\phi)$$



Energy stored
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$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$ Euler-Lagrange equation

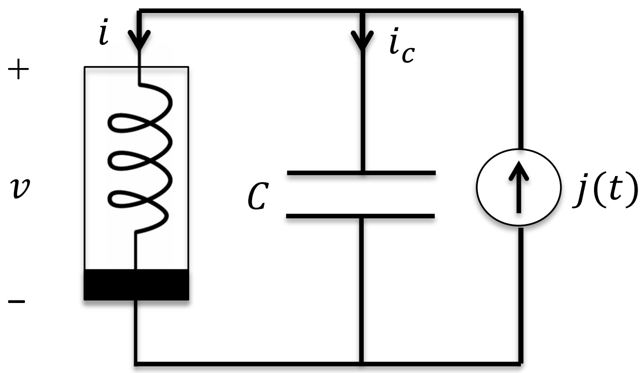
Conjugate momentum $q = \frac{\partial L}{\partial \dot{\phi}}$

As for particles, (ϕ, q) play a very important role:
in the quantum regime they cannot be measured at the same time with the same precision.

Hamiltonian Formulation

Energy stored
in the nonlinear inductor

$$W_I = W_I(\phi)$$



Energy stored
in the capacitor

$$W_c = \frac{1}{2}Cv^2 = \frac{1}{2}C\dot{\phi}^2$$

As for particles, we require a model that treats flux and conjugate momentum on equal footing.

Lagrangian $L\left(\phi, \dot{\phi}; t\right) = \frac{1}{2}C\dot{\phi}^2 - W_I(\phi) + j(t)\phi$

Conjugate momentum $q = \frac{\partial L}{\partial \dot{\phi}} = C\dot{\phi} \Longrightarrow \dot{\phi} = \frac{1}{C}q$

contribution of the current generator

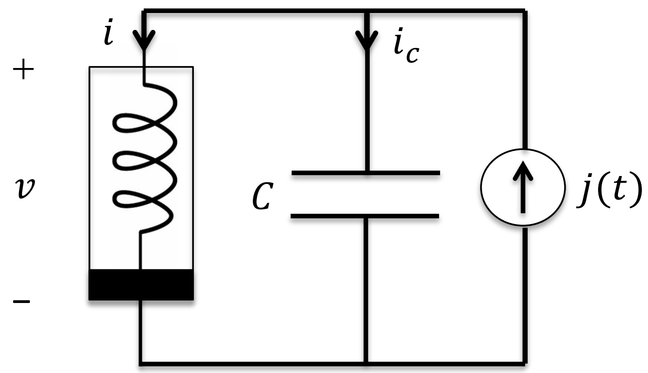
Hamiltonian of the system $H(q, \phi; t) = \frac{1}{2C}q^2 + W_I(\phi) - j(t)\phi$

Energy
stored in the circuit

Hamiltonian Formulation

Energy stored
in the nonlinear inductor

$$W_I = W_I(\phi)$$



Energy stored
in the capacitor

$$W_c = \frac{1}{2} C v^2 = \frac{1}{2} C \dot{\phi}^2$$

Hamiltonian of the system $H(q, \phi; t) = \frac{1}{2C} q^2 + W_I(\phi) - j(t)\phi$

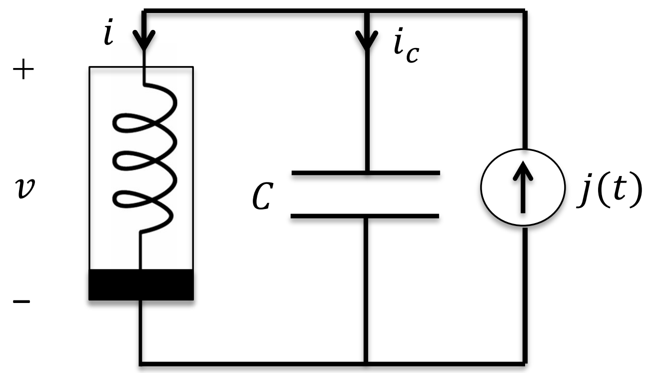
Fundamental property of the Hamiltonian

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Hamiltonian Formulation

Energy stored
in the nonlinear inductor

$$W_I = W_I(\phi)$$



Energy stored
in the capacitor

$$W_c = \frac{1}{2}Cv^2 = \frac{1}{2}C\dot{\phi}^2$$

Hamiltonian of the system $H(q,\phi;t) = \frac{1}{2C}q^2 + W_I(\phi)$

Fundamental property of the Hamiltonian

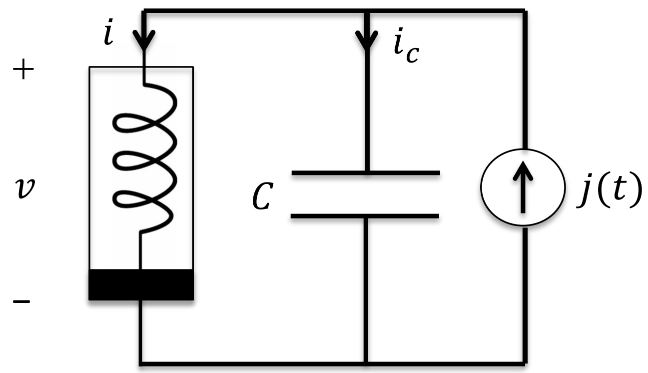
$$\frac{dH}{dt} = 0 \text{ if } j(t) = 0$$

When $j(t) = 0$ the Hamiltonian is a constant of motion: in this case it coincides with the energy stored in the circuit, which is conserved because the circuit is isolated.

Hamiltonian Formulation

Energy stored
in the nonlinear inductor

$$W_I = W_I(\phi)$$



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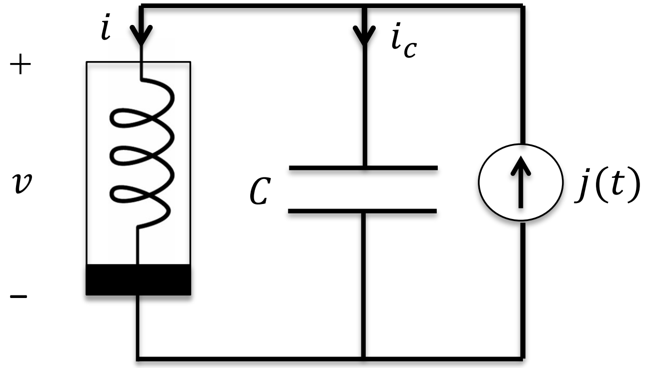
Hamiltonian of the system $H(q,\phi;t) = \frac{1}{2C}q^2 + W_I(\phi) - j(t)\phi$

As for the particle, the Hamiltonian plays a fundamental role when the circuit operate in the quantum regime. The quantum state of the circuit is governed by Schrödinger equation, in which the Hamiltonian pays a fundamental rule.

Hamiltonian Formulation

Energy stored
in the nonlinear inductor

$$W_I = W_I(\phi)$$



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Hamiltonian of the system $H(q, \phi; t) = \frac{1}{2C} q^2 + W_I(\phi) - j(t)\phi$

Hamilton equations
$$\begin{cases} \dot{\phi} = \dfrac{\partial H}{\partial q}, \\ \dot{q} = -\dfrac{\partial H}{\partial \phi}. \end{cases}$$

$$\begin{cases} \dot{x} = \dfrac{1}{m_e} p, \\ \dot{q} = -\dfrac{dW_I}{d\phi} + j(t) \end{cases}$$

State equation
of the circuit

Electron in 1D Potential Well Versus Superconducting Circuit

Classic Electron in 1D potential well

Canonically conjugate variables (x, p)

$$H(p, x; t) = \frac{1}{2m_e} p^2 + W_a(x) - F(t)x$$

Hamilton Equations

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial x} \end{cases}$$

Classic Superconducting circuit

Canonically conjugate variables (ϕ, q)

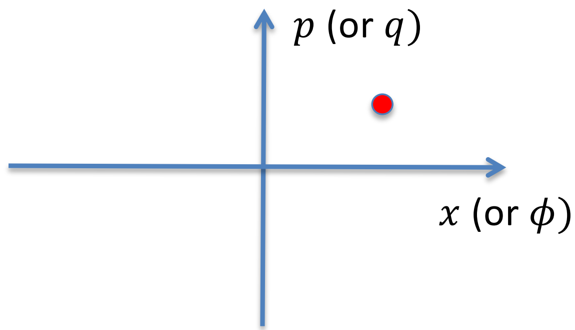
$$H(q, \phi; t) = \frac{1}{2C} q^2 + W_I(\phi) - j(t)\phi$$

Hamilton Equations

$$\begin{cases} \dot{\phi} = \dfrac{\partial H}{\partial q} \\ \dot{q} = -\dfrac{\partial H}{\partial \phi} \end{cases}$$

State in Classical Physics

The set of canonically conjugated variables (x,p) or (ϕ,q) describe the **state** of the system.



i. The state space is a linear vector space.

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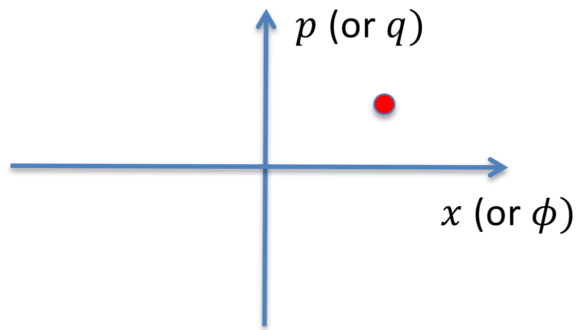

i. The state space is a linear vector space.

ii. The state variables can be measured simultaneously with infinite precision.

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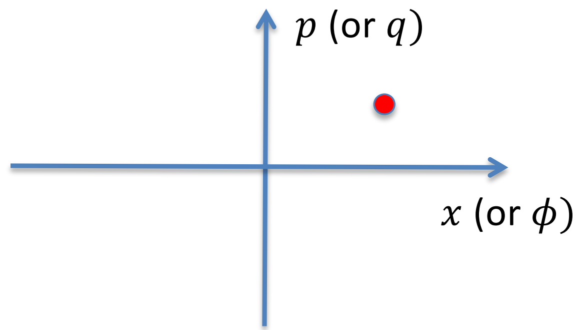

i. The state space is a linear vector space.

ii. The state variables can be measured simultaneously with infinite precision.

iii. The specification of the state variables at any time t uniquely determines the values of all the physical variables of the system at the same time: they are fundamental physical variables of the system.

State in Classical Physics

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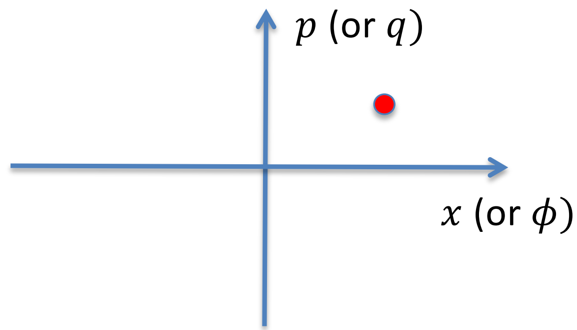


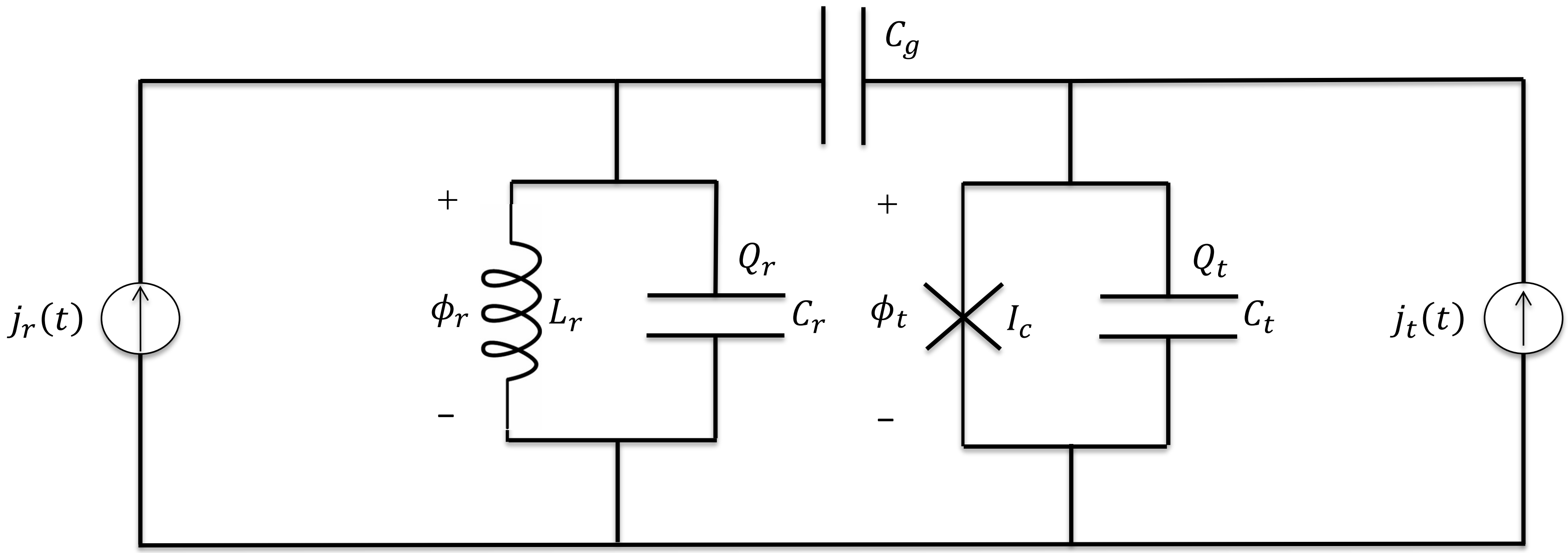
i. The state space is a linear vector space.

ii. The state variables can be measured simultaneously with infinite precision.

iii. The specification of the state variables at any time t uniquely determines the values of all the physical variables of the system at the same time: they are fundamental physical variables of the system.

iv. The evolution of the state is deterministic.

Circuit with Two Degrees of Freedom

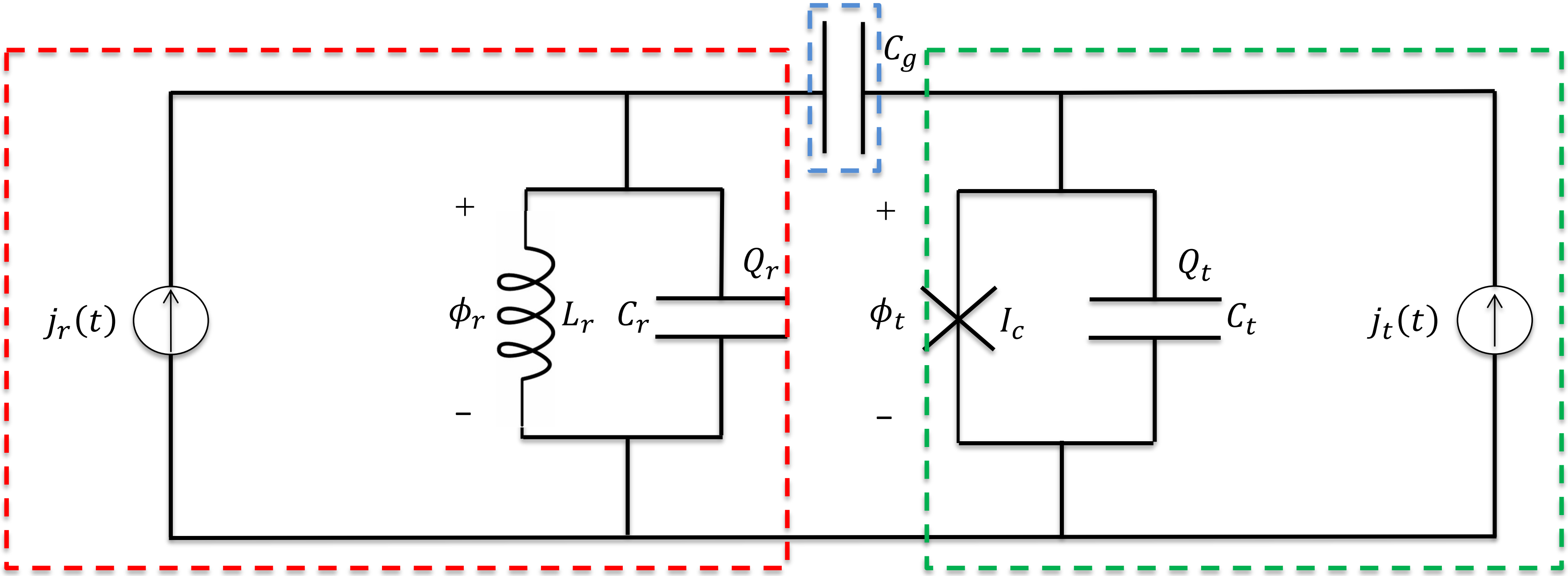


This system has two degrees of freedom: we chose ϕ_r and ϕ_t as degrees of freedom.

S.E. Rasmussen et al., **Superconducting Circuit Companion—an Introduction with Worked Examples,** PRX Quantum **2**, 040204, **2021.**

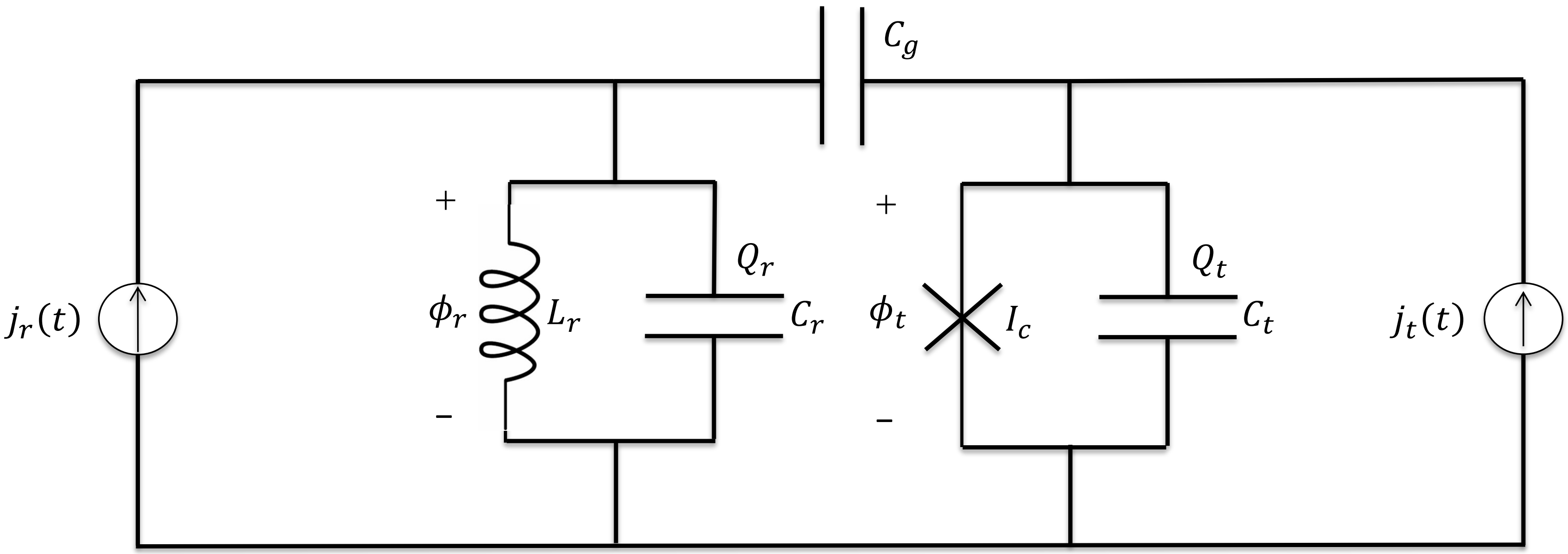
A. Ciani, D. P. DiVincenzo, B. M. Terhal, **Lecture Notes on Quantum Electrical Circuits, 2024,** arXiv:2312.05329.

Lagrangian



$$L(\phi_r, \dot{\phi}_r; \phi_t, \dot{\phi}_t; t) = \left[\frac{1}{2} C_r \dot{\phi}_r^2 - \frac{1}{2L_r} \phi_r^2 + j_r(t)\, \phi_r\right] + \left[\frac{1}{2} C_t \dot{\phi}_t^2 - W_J(\phi_t) + j_t(t)\, \phi_t\right] + \frac{1}{2} C_g (\dot{\phi}_r - \dot{\phi}_t)^2$$

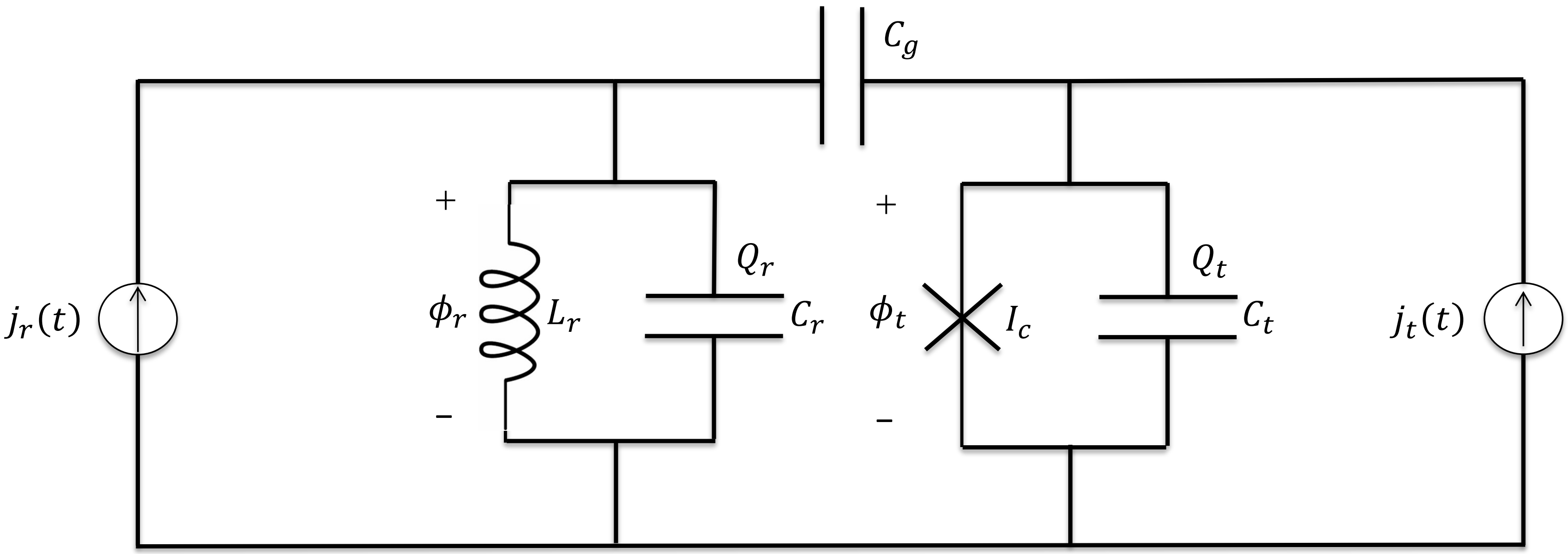
Canonically Conjugate Variables



$$\begin{cases} q_r = \dfrac{\partial L}{\partial \dot{\phi}_r} = \left(1 + \dfrac{C_g}{C_r}\right) Q_r - \dfrac{C_g}{C_t} Q_t \\ q_t = \dfrac{\partial L}{\partial \dot{\phi}_t} = -\dfrac{C_g}{C_r} Q_r + \left(1 + \dfrac{C_g}{C_t}\right) Q_t \end{cases}$$

q_r is canonical conjugate to ϕ_r, and q_t is canonical conjugate to ϕ_t.

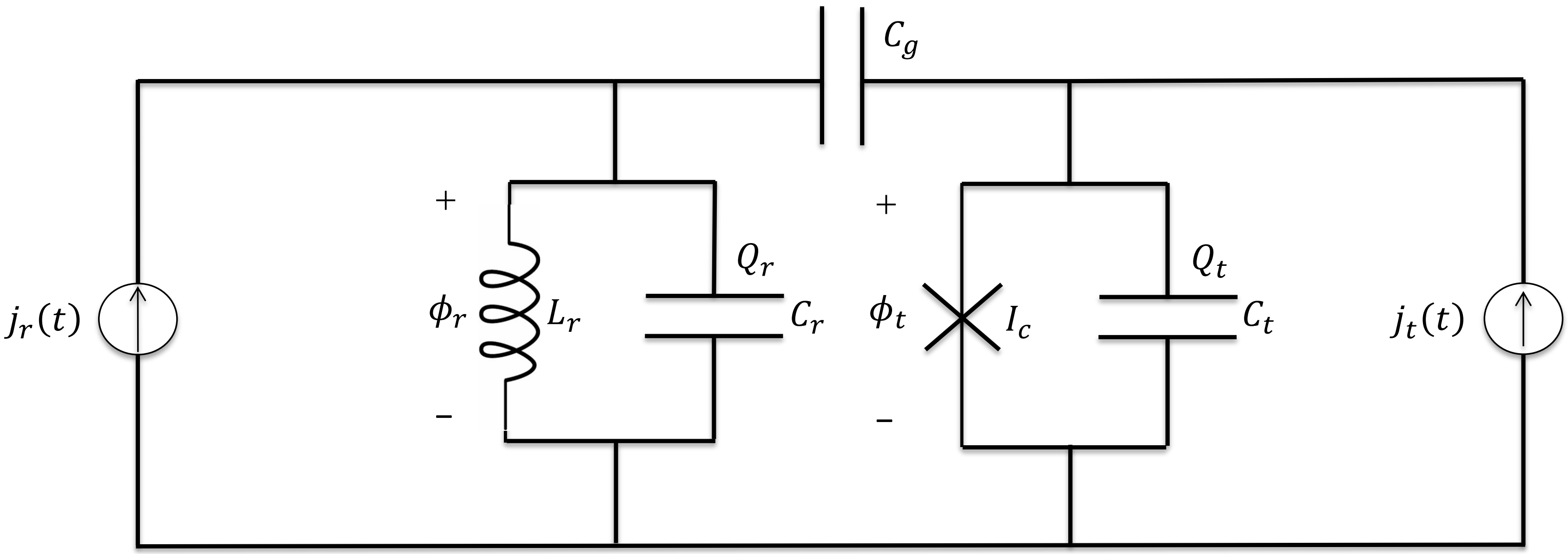
Capacitive two – ports potential coefficients



$$p_r = \frac{C_g + C_r}{C_*^2}, p_t = \frac{C_g + C_t}{C_*^2}, p_m = \frac{C_g}{C_*^2}$$

$$C_*^2 = C_g C_r + C_r C_t + C_g C_t$$

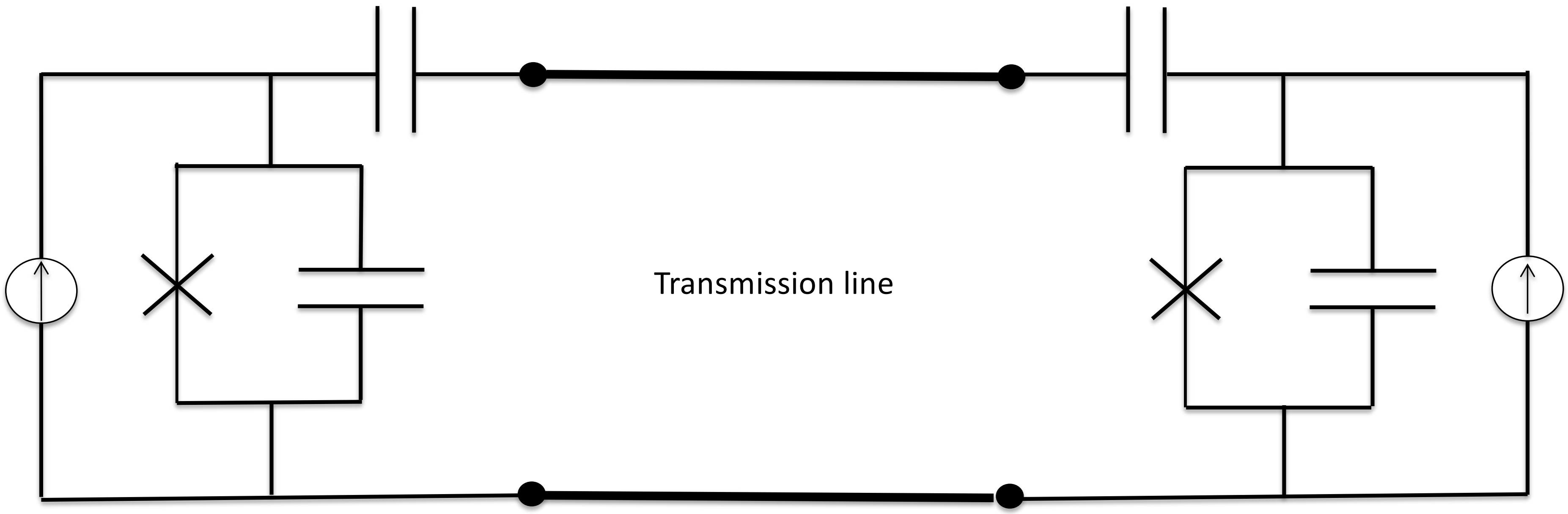
Hamiltonian



$$H(q_r, \phi_r; q_t, \phi_t; t) = \frac{q_r^2}{2C_{rr}} + \frac{1}{2L_r}\phi_r^2 + \frac{q_t^2}{2C_{tt}} + E_J[1 - cos(2\pi\phi_t/\Phi_0)] + \frac{q_r\ q_t}{C_m} - (j_r\phi_r + j_t\phi_t)$$

$$C_{rr} = 1/p_r,\ E_J = I_c\Phi_0\ /2\pi,\ C_{tt} = 1/p_t,\ C_m = \frac{1}{p_m}$$

Circuit with Transmission Lines



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