Introduction to Circuit Quantum Electrodynamics

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Napoli, February 3-7 2025

Scuola Nazionale Dottorandi di Elettrotecnica “Ferdinando Gasparini”: XXVII Stage

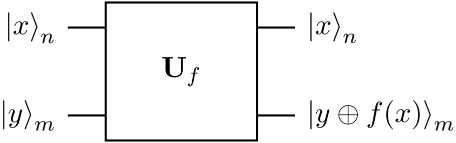
INTERNATIONAL YEAR OF
Quantum Science
and Technology

Outline (1/2)

Outline (2/2)

3. Noise and Decoherence
   - Types of Noise: Systematic vs Stochastic Noise
   - Longitudinal relaxation
   - Pure dephasing
   - Evolution of lifetimes and coherence times in superconducting qubits

The general computational process



Why do we need to have separate input and output register?

We have to guarantee the reversibility of our transformation even in presence of non-injective functions

The introduced transformation is invertible. Indeed, $\mathbf{U}_f$ is its own inverse:

$$\mathbf{U}_f \mathbf{U}_f |x\rangle_n |y\rangle_m = \mathbf{U}_f |x\rangle_n |y \oplus f(x)\rangle_m = |x\rangle_n |y \oplus f(x) \oplus f(x)\rangle_m = |x\rangle_n |y\rangle_m$$

Hadamard Gate

$$\mathbf{H}\,|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \qquad \mathbf{H}\,|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\mathbf{H}\otimes\mathbf{H}\,|0\rangle\otimes|0\rangle = \left(\mathbf{H}\,|0\rangle\right)\left(\mathbf{H}\,|0\rangle\right) = \frac{1}{2}\left(|0\rangle\,|0\rangle + |1\rangle\,|0\rangle + |0\rangle\,|1\rangle + |1\rangle\,|1\rangle\right)$$

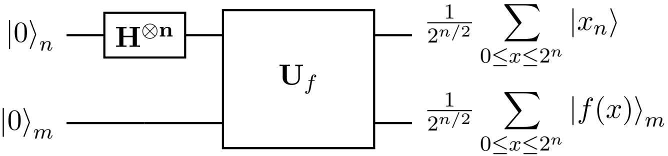
$$\mathbf{H}^{\otimes n}\,|0\rangle_n = \frac{1}{2^{n/2}}\sum_{0\leq x\leq 2^n}|x\rangle_n$$

$|0\rangle_n$

$$\frac{1}{2^{n/2}}\sum_{0\leq x\leq 2^n}|x_n\rangle$$

Quantum Parallelism?

$$\mathbf{U}_f\left(\mathbf{H}^{\otimes n}\otimes\mathbf{1}_m\right)|0\rangle_n|0\rangle_m=\frac{1}{2^{n/2}}\sum_{0\leq x\leq 2^n}\mathbf{U}_f\left(|x\rangle_n|0\rangle_m\right)=\frac{1}{2^{n/2}}\sum_{0\leq x\leq 2^n}|x\rangle_n|f(x)\rangle_m$$



- before letting $\mathbf{U}_f$ act, we apply a Hadamard transformation to every Qbit of the input register, initially in the state $|0\rangle_n$,
- the result of the computation is described by a state whose structure cannot be explicitly specified without knowing the result of all 2^n evaluations of $f(x)$

Quantum Parallelism?

- If we have a hundred Qbits in the input register, initially all in the state $|0\rangle_{100}$ (and m more in the ouput register)
- if a hundred Hadamard gates act on the input register before the application of $\mathbf{U}_f$, then the form of the final state contains the results of $2^{100} \approx 10^{30}$ evaluations of the function $f(x)$
- A billion billion trillion evaluations! **quantum parallelism**

Caveat

There is no way to know what the state is!

Quantum Parallelism?

$$\mathbf{U}_f\left(\mathbf{H}^{\otimes n}\otimes\mathbf{1}_m\right)|0\rangle_n|0\rangle_m=\frac{1}{2^{n/2}}\sum_{0\leq x\leq 2^n}\mathbf{U}_f\left(|x\rangle_n|0\rangle_m\right))=\frac{1}{2^{n/2}}\sum_{0\leq x\leq 2^n}|x\rangle_n|f(x)\rangle_m$$

- we send all $n+m$ Qbits through measurement gates
- the Born rule tells us that if the state of the registers has the above form then with equal probability the result of measuring the Qbits will be one of the values of x less than 2^n
- while the results of measuring the Qbits in the output register will be the value of f for that particular x
- After the measurement the state of the register reduces to $|x_0\rangle|f(x_0)\rangle$

No cloning theorem

If there were an easy way to make copies of the output state prior to make the measurement, without running the computation again, then one could with high probability, learn the value of $f(x)$ for several different values of x

No cloning theorem

There is no unitary transformation that can take the state $|\psi\rangle_n |0\rangle_n$ into the state $|\psi\rangle_n |\psi\rangle_n$ for arbitrary $|\psi\rangle_n$.

No cloning theorem

proof

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Let us assume

$$\mathbf{U}_f |\psi\rangle |0\rangle = |\psi\rangle |\psi\rangle$$
$$\mathbf{U}_f |\varphi\rangle |0\rangle = |\varphi\rangle |\varphi\rangle$$

From **linearity** we expect that

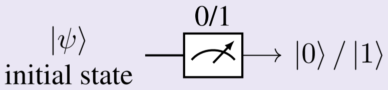
$$\mathbf{U}_f (a |\psi\rangle + b |\varphi\rangle) |0\rangle = a |\psi\rangle |\psi\rangle + b |\varphi\rangle |\varphi\rangle$$

By applying the **definition** we have

$$\mathbf{U}_f (a |\psi\rangle + b |\varphi\rangle) |0\rangle = (a |\psi\rangle + b |\varphi\rangle)(a |\psi\rangle + b |\varphi\rangle)$$
$$= a^2 |\psi\rangle |\psi\rangle + b^2 |\varphi\rangle |\varphi\rangle + ab |\psi\rangle |\varphi\rangle + ba |\varphi\rangle |\psi\rangle$$

Take home message

One can extract useful information about relations between the values of f for several values of x, which classical computer can only get by making independent evaluations



$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \quad \text{with} \quad |\alpha_0|^2 + |\alpha_1|^2 = 1$$

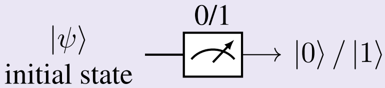
Born rule

If the state of the Q-bit is the superposition of the states $|0\rangle$ and $|1\rangle$ with amplitudes α_0 and α_1 then the result of the measurement is

- 0 with probability $|\alpha_0|^2$
- 1 with probability $|\alpha_1|^2$

Measurement Gate

Collapse of the state



$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \quad \text{with} \quad |\alpha_0|^2 + |\alpha_1|^2 = 1$$

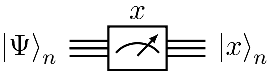
State collapse

What is the state of the Q-bit after the measurement?

- if the measurement returns the value 0, the state will be $|0\rangle$;
- if the measurement returns the value 1, the state will be $|1\rangle$

Measurement Gate

Born rule and collapse of the state



If the state of n Qbits is

$$|\Psi\rangle_n = \sum_{0 \leq x < 2^n} \alpha_x |x\rangle_n$$

Born rule

the probability that the zeros and the ones resulting from measurement of all the Q-bits will give the binary expansion of the integer x is

$$p(x) = |\alpha_x|^2 .$$

Collapse of the state

If the display of the measurement gate indicates x, then the Q-bits emerging from that measurement gate is in the classical-basis state $|x\rangle$

Period finding

$$f(x) = 7^x \mod 15$$

- $7^1 = 7 \mod 15 = 7$
- $7^2 = 49 \mod 15 = (3 \times 15 + 4) \mod 15 = 4$
- $7^3 = 343 \mod 15 = (22 \times 15 + 13) \mod 15 = 13$
- $7^4 = 2401 \mod 15 = (160 \times 15 + 1) \mod 15 = 1$
- $7^5 = 16807 \mod 15 = (1120 \times 15 + 7) \mod 15 = 7$
- $7^6 = 117649 \mod 15 = (7843 \times 15 + 4) \mod 15 = 4$
- $7^7 = 823543 \mod 15 = (54902 \times 15 + 13) \mod 15 = 13$
- $\ldots$

Quantum period finding: preliminary remarks

- We can **crack the RSA code** if we have a way to find the period r of the periodic function

$$f(x) = b^x \pmod N$$

- $b^x \pmod N$ is the kind of function whose values within a period hop about so irregularly as to offer no obvious clues about the period.

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- One could try evaluating $f(x)$ for random x until one found two different values of x for which f agreed.
- Those values would differ by a multiple of the period, which would provide some important information about the value of the period itself.
- But this is an **inefficient** way to proceed, even classically.

Quantum period finding: preliminary remarks

- To have an appreciable probability of finding r by random searching requires a **number of evaluations of f that is exponential in n_0**

- There are classical ways to improve on random searching, using, for example, Fourier analysis, but no classical approach is known that does not require a time that grows faster than any power of n_0.

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- To have an appreciable probability of finding r by random searching requires a **number of evaluations of f that is exponential in n_0**
- There are classical ways to improve on random searching, using, for example, Fourier analysis, but no classical approach is known that does not require a time that grows faster than any power of n_0.
- With a quantum computer, however, quantum parallelism gets us tantalizingly close to solving the problem with a single application of $\mathbf{U}_f$, and enables us to solve it completely with probability arbitrarily close to unity in a time that grows only as a **low-order polynomial in n_0**.

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- To deal with values of x and $f(x) = b^x (\bmod N)$ between 0 and N, both the input and output registers must contain at least n_0 Qbits.

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- Doubling the number of Qbits in the input register ensures that the range of values of x for which $f(x)$ is calculated contains at least N full periods of f.
- This redundancy turns out to be essential for a successful determination of the period by Shor's method.

Quantum period finding: preliminary remarks

- We construct the state

$$\frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle_n |f(x)\rangle_{n_0}$$

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- x_0 is the smallest value of x $(0 \leq x_0 < r)$ for which $f(x_0) = f_0$
- m is the smallest integer for which $mr + x_0 \geq 2^n$

$$m = \left[\frac{2^n}{r}\right] \text{ or } m = \left[\frac{2^n}{r}\right] + 1,$$

depending on the value of x_0 (where $[x]$ is the integral part of x - the largest integer less than

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$$|\Psi\rangle_n = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + kr\rangle_n \, .$$

- if we could have a small number of **identical copies** of the state $|\Psi\rangle_n$ the job would be done
- a measurement in the computational basis would yield a random one of the values $x_0 + kr$
- and the difference between the results of pairs of measurements on such identical copies would give us a collection of random multiples of r from which r itself could straightforwardly be extracted.

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- and the difference between the results of pairs of measurements on such identical copies would give us a collection of random multiples of r from which r itself could straightforwardly be extracted.
- But this possibility is ruled out by the **no-cloning theorem**.
- We can only extract **a single value** of $x_0 + kr$ for unknown random x_0, which is useless for determining r.

The quantum Fourier transform

The n-Qbit quantum Fourier transform is defined as:

$$\mathbf{U}_{\mathrm{FT}}|x\rangle_n = \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i xy/2^n} |y\rangle_n .$$

- The product xy is **ordinary multiplication**
- One easily verifies that $\mathrm{U}_{\mathrm{FT}}|x\rangle$ is is unitary.

Quantum Fourier Transform

A diagram of a circuit that illustrates, for four Qbits, the construction of the quantum Fourier transform $\mathbf{U}_{\mathrm{FT}}$

Finding the period

$$|\Psi\rangle_n = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + k\, r\rangle_n$$

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- To get information about r we apply the quantum Fourier transformation to the input register:

$$\mathbf{U}_{\mathrm{FT}} \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + kr\rangle = \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (x_0+kr) y/2^n} |y\rangle$$

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- The factor $e^{2\pi i x_0 y/2^n}$, in which x_0 explicitly occurs, drops out of this probability, and we get

$$p(y) = \frac{1}{2^n m} \left| \sum_{k=0}^{m-1} e^{2\pi i k r y/2^n} \right|^2 .$$

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- The probability $p(y)$ is a simple explicit function of the integer y
- $p(y)$ has maxima when y is close to integral multiples of $2^n/r$.
- We may have to **repeat the procedure** a small number of times to achieve a high probability of learning the period r.

Finding the period

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- the probability that the measured value of y will be within $\frac{1}{2}$ of an integral multiple of $2^n/r$ is

$$p\left(\left| y - \frac{2^n}{r} \right| < \frac{1}{2} \right) \geq 0.4$$

Finding the period

- We calculate this lower bound for $p(y)$ when

$$y = y_h = h2^n/r + \varepsilon_h \qquad \text{with} \quad |\varepsilon_h| \leq \frac{1}{2}$$

Only the term in ε_h contributes to the exponentials.

- The summation is a geometric series, which can be explicitly summed to give

$$p\left(y_h\right) = \frac{1}{2^n m}\left|\left(\frac{1 - e^{2\pi i m r y_h/2^n}}{1 - e^{2\pi i r y_h/2^n}}\right)\right|^2 = \frac{1}{2^n m}\left|\left(\frac{1 - e^{2\pi i m r \varepsilon_h/2^n}}{1 - e^{2\pi i r \varepsilon_h/2^n}}\right)\right|^2$$

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$$p\left(y_h\right) = \frac{1}{2^n m}\frac{\sin^2\left(\pi\varepsilon_h m r/2^n\right)}{\sin^2\left(\pi\varepsilon_h r/2^n\right)}$$

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- m is within an integer of $2^n / r$
- we can with negligible error replace $mr/2^n$ by 1 in the numerator
- we replace the sine in the denominator by its (extremely small) argument:
- $$p\left(y_h\right) = \frac{1}{2^n m} \left(\frac{\sin\left(\pi \varepsilon_h\right)}{\pi \varepsilon_h r / 2^n}\right)^2 = \frac{1}{r} \left(\frac{\sin\left(\pi \varepsilon_h\right)}{\pi \varepsilon_h}\right)^2$$

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- When $0 < x < \pi/2$, the graph of $\sin x$ lies above the straight line connecting the origin to the maximum at $x = \pi/2$:

$$\sin x \geq x / \left(\frac{1}{2}\pi\right), \quad 0 \leq x \leq \pi/2$$

Finding the period

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- Since there are at least $r-1$ different values of h
- Since r is a large number, one has at least a 40% chance $(4/\pi^2 = 0.4053)$ of getting one a value for y that is within $\frac{1}{2}$ of an integral multiple of $2^n/r$.

Noise and Decoherence

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- the knowledge of the initial state and the Hamiltonian allows us to predict the qubit state at any future time.
- In **opens systems**, the qubit interacts with uncontrolled environmental degrees of freedom.
- These interactions introduce **fluctuations (noise)**, which cause the qubit's evolution to deviate from the ideal prediction.
- Over time, this deviation leads to **decoherence** — the loss of the intended quantum state.

Types of Noise: Systematic Noise

In superconducting quantum computers, the distinction between systematic and stochastic noise is crucial for understanding and mitigating errors.

Systematic noise

- **Source** Arises from predictable imperfections in the hardware or control system, such as fabrication defects in Josephson junctions, crosstalk between qubits, or errors in microwave pulse shaping.
- **Impact**: These errors can cause coherent, repeatable deviations in the qubit states or gate operations.
- **Mitigation**: Calibration and fine-tuning of control parameters. Error mitigation techniques. System-level modeling to identify and correct systematic errors.

Types of Noise : Stochastic Noise

In superconducting quantum computers, the distinction between systematic and stochastic noise is crucial for understanding and mitigating errors.

Stochastic noise

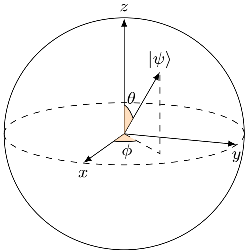
- **Source**: Originates from random processes like thermal fluctuations, quantum decoherence, or background radiation interacting with the qubits.
- **Impact**: Leads to incoherent, unpredictable errors such as spontaneous qubit state flips (bit-flip) or phase changes (phase-flip).
- **Mitigation**: Improving qubit isolation and shielding from external noise. Operating at ultra-low temperatures to reduce thermal noise. Using quantum error correction codes to detect and correct random errors.

Bloch sphere representation of the quantum state of the qubit.

The state of a qubit is represented as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

$$0 < \theta < \pi;\ 0 < \varphi < 2\pi$$

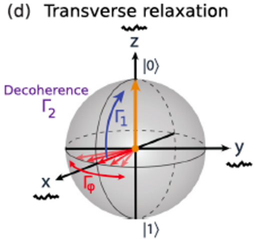


- The z-axis is **longitudinal** in the qubit frame, corresponding to $\hat{\sigma}_z$ terms in the qubit Hamiltonian.
- The x-y plane is **transverse** in the qubit frame, corresponding to $\hat{\sigma}_x$ and $\hat{\sigma}_y$ terms in the qubit Hamiltonian.

Longitudinal relaxation

Energy Exchange and Decoherence

- Longitudinal relaxation results from the energy exchange between the qubit and its environment due to **transverse noise**
- Transverse noise couples to the qubit in the x-y plane and **drives transitions** $|0\rangle \leftrightarrows |1\rangle$:



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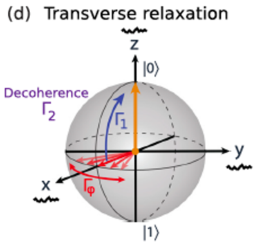

- **Emission (Relaxation):** An excited qubit in $|1\rangle$ can spontaneously emit a photon into its environment and decay to the ground state $|0\rangle$ at a rate $\Gamma_{1\downarrow}$ (blue arrow).
- **Absorption (Excitation):** Conversely, the qubit in $|0\rangle$ may absorb energy (a photon) from the environment and be excited to $|1\rangle$ at a rate $\Gamma_{1\uparrow}$ (orange arrow).

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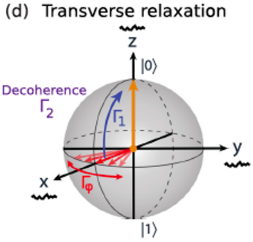

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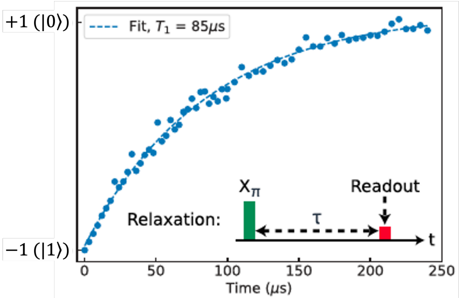
- In the typical operating regime $k_B T \ll \hbar\omega_q$ $(T = 20mK$ and $\frac{\omega_t}{2\pi} \approx 5\mathrm{GHz})$ the up-rate is suppressed. This fact leads to the overall decay rate $\Gamma_1 \cong \Gamma_{1\uparrow}$.

Measuring the decay lifetime

Protocol

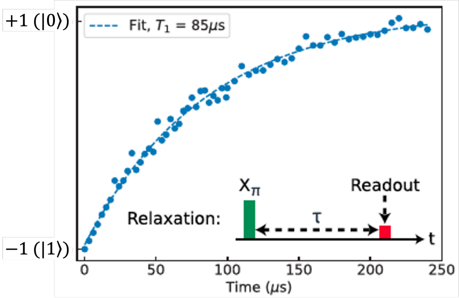
- **Prepare** the qubit in the ground state $|0\rangle$;
- At time $t = 0$, apply a **qubit flip** $\mathbf{X}_\pi$ operation to prepare the qubit in the excited state $|1\rangle$.

$T_1 = 1/\Gamma_1$



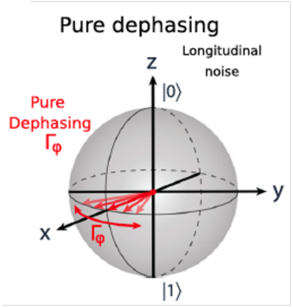
- **Prepare** the qubit in the ground state $|0\rangle$;
- At time $\mathrm{t} = 0$, apply a **qubit flip** $\mathbf{X}_\pi$ operation to prepare the qubit in the excited state $|1\rangle$.
- **Wait a variable time**, τ, and then **measure** the qubit in the $\{|0\rangle, |1\rangle\}$ basis;
- For each value τ, this procedure is repeated to obtain an ensemble average of the qubit polarization:

$T_1 = 1/\Gamma_1$



Pure dephasing

- Dephasing is a **non-dissipative decoherence process** (no energy is exchanged with the environment).
- It is often driven by a **non-resonant** or dispersive interaction with the environmental modes.

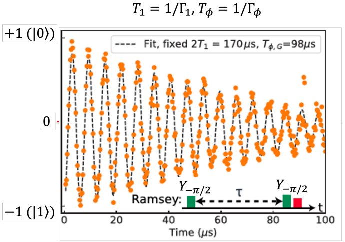


- Effectively, dephasing can usually be described as a coupling between the qubit phase and the environment.
- Instead of random emission events (as in decay), dephasing can be thought of as random phase kicks on the qubit system, induced by dispersive environmental coupling.

A protocol for measuring the dephasing lifetime

A protocol for measuring the dephasing lifetime is the following (Ramsey measurement protocol) :

- Prepare the qubit in the ground state, $|0\rangle$;
- At time $\mathrm{t} = 0$, prepare the qubit in the superposition state $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ by applying the $Y_{-\pi/2}$ transformation;
- Allow the state to evolve freely for a variable time, τ, and then measure the state in the $\{|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$,
- Repeat many times for each value of t to calculate the the probability of finding the qubit in the $|+\rangle$ state.



Evolution of lifetimes and coherence times in superconducting qubits

- *JJ-based qubits* are qubits based on Josephson junction.
- *Bosonic encoded qubits* are qubits where the quantum information is encoded in superpositions of multi-photon states in a quantum harmonic oscillator, and a Josephson junction circuit mediates qubit operation and readout
- *Error corrected qubits* represent qubit encodings in which a layer of active error-correction has been implemented to increase the encoded qubit lifetime.